Stochastic volatility Libor modeling and efficient algorithms for optimal stopping problems

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit verschiedenen Aspekten der Finanzmathematik. Ein allgemeiner Rahmen wird erarbeitet in welchem stochastische Prozesse existieren mit deren Hilfe Finanzinstrumente modelliert werden können. Die risikoneutrale Bewertung solcher Instrumente wird erläutert, wobei der Schwerpunkt auf Zinsvorwärtsraten, sogenannten Liboren liegt.

Danach wird ein erweitertes Libor Markt Modell betrachtet, welches genug Flexibilität bietet um akkurat an verschiedene Marktdaten von Caplets und Swaptions zu kalibrieren. Dafür werden analytische Preisformeln für solche Instrumente für eine effiziente Kalibrierung benötigt. In diesem Modell erhalten wir solche Preisformeln nach einer Approximation unter Zuhilfenahme des Konzeptes der Schnellen Fourier Transformierten.

Weiterhin sind wir interessiert an der Bewertung komplexerer Finanzderivate, zum Beispiel durch Simulation. In einem Modell mit hohen Dimensionen können solche Simulationen sehr zeitaufwendig sein. Wir zeigen mögliche Verbesserungen bezüglich der Komplexität der Simulation auf. Faktorreduktion identifiziert die Haupfaktoren des Modells und kann zu einer signifikanten Komplexitätsverringerung führen. Zusätzlich erweitern wir ein Simulationsschema, eingeführt in Andersen [2007], von einer auf mehrere Dimensionen, wobei wir das Konzept des „Momentmatchings“ zur Approximation des Volaprozesses in einem Heston Modell nutzen. Daraus resultiert eine verbesserte Konvergenz des Gesamtprozesses und somit eine verringerte Komplexität bei der simulationsbasierten Bewertung von Optionen.

Als Nächstes beschäftigen wir uns mit der Bewertung sogenannter Amerikanischer Optionen. Diese können als Stoppproblem interpretiert werden. In höheren Dimensionen, wie sie zum Beispiel bei Basketauszahlungsprofilen auftreten, ist die simulationsbasierte Bewertung meist die einzig praktikable Lösung, da diese eine dimensionsunabhängige Konvergenz gewährleistet. Eine neue Methode der Varianzreduktion, die Multilevel-Idee, wird hier auf die simulationsbasierte Optionsbewertung angewandt. Wir leiten eine untere Preisschranke unter zu Hilfenahme der Methode der „policy iteration“ her. Dafür werden Konvergenzraten für die Simulation des Optionspreises erarbeitet und eine detaillierte Komplexitätsanalyse dargestellt.

Abschließend wird das Preisen von Amerikanischen Optionen unter Modellunsicherheit behandelt, wodurch die Restriktion, nur ein bestimmtes Wahrscheinlichkeitsmodell zu betrachten, entfällt. Verschiedene Modelle können plausibel sein und zu verschiedenen Optionswerten führen. Dieser Ansatz führt zu einem nichtlinearen, verallgemeinerten Erwartungsfunktional. Wir erhalten eine verallgemeinerte Snell’sche Einhüllende und leiten das Bellman Prinzip her. Dadurch kann eine Lösung durch Rückwärtsrekursion erhalten werden. Unser numerischer Algorithmus liefert untere und obere Preisschranken.

Abstract

The work presented here deals with several aspects of financial mathematics. A general framework is established in which the existence of stochastic processes can be guaranteed which can be used to model financial instruments. A risk neutral evaluation of such instruments is explained, while focusing on the modeling of interest forward rates, so called Libors. These allow to work without the usual assumption of the existence of a risk neutral financial instrument.

Next we deal with an extended Libor market model offering enough flexibility to accurately calibrate to various market data for caplets and swaptions. To compete with standard models with regard to calibration, analytical price formulas for European options within this model are called for. Those are obtained after an approximation using the concept of Fast Fourier Transform.

Further we are interested in the evaluation of more complex financial derivatives for instance by simulation. Due to the typically high dimension of the model involved such simulations can be very time consuming. We show possible improvements regarding the complexity of the simulation. Factor reduction, a method known from statistics, identifies the main driving factors of a model and may lead to a significant reduction of complexity. In addition we extend a known simulation scheme, established in Andersen [2007], from one to multiple dimensions using the concept of moment matching for the approximation of the vola process in a Heston model. This results in an improved convergence of the whole process thus reducing the complexity of a simulation based evaluation of options.

Next we address the problem of evaluating so called American options. These vastly traded options can be interpreted as a stopping problem. An efficient evaluation of these options, particularly in high dimensions, is a delicate problem. For example when dealing with basket cash flows, a simulation based approach offering dimension independent convergence often happens to be the only practicable solution. A new method of variance reduction given by the multilevel idea is applied to this approach. Starting with the ideas of Belomestny and Schoenmakers [2011] a lower bound for the option price is obtained using “multilevel policy iteration” method. Convergence rates for the simulation of the option price are obtained and a detailed complexity analysis is presented.

Finally we deal with the valuation of American options under model uncertainty. This lifts the restriction of considering one particular probabilistic model only. Different models might be plausible and may lead to different option values. This approach leads to a non-linear expectation functional, calling for a generalization of the standard expectation case. We obtain a generalized Snell envelope, enabling a backward recursion via Bellman principle. We then provide a numerical algorithm to valuate American options under ambiguity, obtaining lower and upper price bounds.

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Contents

Background and overview

The work presented here deals with several branches of financial mathematics. To begin with, an important branch is the area of interest rates. One important class of interest rates is given by the so called London Interbank Offer Rate (Libor). The Libor gained bad publicity recently due to the manipulation scandal in 2012 and its aftermath. Nevertheless it still has big impact on the financial industry. The Libor modeling framework was developed almost two decades ago simultaneously by Miltersen, Sandmann and Sondermann [1997], Brace, Gatarek and Musiela [1997] and Jamshidian [1997], where in the latter approach, based on the concept of a state-price deflator (see also Duffie [1992]), existence of a risk-free asset was not required. For a detailed overview, see also Andersen and Piterbarg [2010], Brigo and Mercurio [2001] and Schoenmakers [2005]. The Libor framework is still considered to be the universal tool for evaluation of structured interest rate products because it offers great flexibility for including different sources of randomness, such as Brownian motions, Lévy processes or even more general semimartingales. Further, these sources of randomness may be linked with various types of volatility structures, e.g. stochastic volatility, local volatility, or deterministic volatilities. Nevertheless, designing a Libor model that can be calibrated in a feasible way to an (in some sense) complete set of liquid market quotes (e.g. caps and swaptions for different strikes and different maturities) remains a perennial problem. The early works considered Libor models driven by a set of Brownian motions equipped with some deterministic volatility structure. Those so called market models where quiet popular as they allowed for analytical cap(let) pricing and (approximate) analytic swaption pricing via Black 76 type formulas. The drawback of these so called Libor market models is that they are not able to match implied volatility "smile/skew" behavior observed in the cap and swap markets however. Moreover, these "smile/skew" effects became even more pronounced over the years.

In the past several proposals in order to capture "smile/skew" behavior have been made. One line of research concentrates on Libor models driven by compound Poisson processes as in Glasserman and Kou [2003] and Belomestny and Schoenmakers [2006] or even infinite activity Lévy processes like in Eberlein and Özkan [2005]. Further we mention the Constant Elasticity of Variance (CEV) based extension of the Libor market model by Andersen and Andreasen [2000] and the displaced diffusion Libor market model by Joshi and Rebonato [2001]. Brigo and Mercurio propose in Brigo and Mercurio [2001] a local volatility model consistent with a mixture of log-normal transition densities and some variations on this. In another work by Wu and Zhang [2006] a Heston version of the Libor market model is proposed. In the dynamics of this model, which is related to the models in Piterbarg [2004] and Andersen and Brotherton-Ratcliffe [2001], the volatility of each forward Libor contains a common stochastic volatility factor $\sqrt{v}$ where v is a

Cox-Ingersoll-Ross type square-root process, correlated with the Libor driving Brownian motions. Moreover, Wu and Zhang [2006] shows that their model has strong potential to produce smiles and skews (in particular due to the correlated v), and they present Fourier based quasi analytic approximation methods for the pricing of caps and swaptions. As a further extension Belomestny, Mathew and Schoenmakers [2011] proposed a multi-factor stochastic volatility model where each Libor gets his own volatility process v_i.

Apart of modeling and calibration, the evaluation of complexly structured financial derivatives is of main importance. Such products can usually not be evaluated in closed form. A canonical alternative is Monte Carlo simulation, especially in higher dimensional structures as in the case of a Libor system. The most prominent non-plain vanilla option is the early exercise option, also termed American or Bermudan option.

In contrast to European options that can be exercised only at a fixed point in time, American options give the owner the right to exercise ones over a whole time interval. The pricing of American options involves the solution of an optimal stopping problem of the form

$$V(0) := \sup_{\tau\in[0,T]} \mathbb{E}\left[X(\tau)\right].$$

Here τ is an arbitrary stopping time taking values in $[0,T]$ and $(X(t))_{0\leq t\leq T}$ is some adapted cash-flow process. One thus aims to find the expected value $V(0)$ from exercising optimally. By considering the above stopping problem from a generic starting point t rather than $t=0$, one obtains the so called Snell envelope. For a discrete set of exercise times, i.e. Bermudan style options as assumed in this sequel, the Snell envelope satisfies a Bellman principle. By this principle it is possible to characterize the evolution of the Snell envelope, hence the (discounted) option value, by a backward dynamic program. The first breakthrough approaches to efficiently price American/Bermudan options were regression based Monte Carlo methods introduced by Carriere [1996], Longstaff and Schwartz [2001] and Tsistsiklis and Van Roy [2000]. Other methods related to a recursive representation of the Snell envelope (Bellman principle) are, for example, random tree method of Broadie, Glasserman and Ha [2004], the stochastic mesh method of Broadie and Glasserman [2004], and the quantization algorithm by Bally and Pages [2003]. We further mention Kolodko and Schoenmakers [2006], who considered a class of policy iterations while in Bender, Kolodko and Schoenmakers [2008] it is demonstrated that the latter approach can be effectively combined with the Longstaff-Schwartz approach. Another particular approach in this line is to search for a suitable parametric family of exercise boundaries and then maximize the solutions of the corresponding family of boundary value problems over the parameters. This concept has been applied in the context of Bermudan swaptions in a Libor market model in Andersen [2000]. The aforementioned algorithms rely on simulation of an approximated optimal stopping time and thus give naturally rise to a lower biased approximation of the problem. Therefore these approaches are called primal.

In contrast to the primal approach a novel dual method was established by Davis and Karatzas [1994], Rogers [2002] and Haugh and Kogan [2004]. Instead of maximizing over stopping times, in the dual approach one minimizes over a set of martingales $(M(t))_{0\leq t\leq T}$

starting in zero. That is, the solution of the stopping problem is represented by

$$V(0) = \inf_{M} \mathbb{E}\left[\sup_{0\leq t\leq T}\{X(t) - M(t)\}\right].$$

Since the option price is obtained as an infimum over martingales one thus obtains an upper biased approximation of the solution.

The Libor model studied in this work involves a Heston type volatility structure. Therefore the simulation of Heston type SDEs is of prime importance. It is well known that the usage of a simple Euler scheme would generally fail, as the discretized square-root process can become negative. A simple adaption was presented in Lord, Koekkoek and van Dijk [2010], by using a full truncation ansatz, prohibiting negative values. Nevertheless the use of the Euler scheme leaves us with poor convergence behavior. Some recent approaches to an efficient discretization of the continuous-time Heston dynamics for purposes of Monte Carlo simulation in a one-dimensional setting where made by Kahl and Jackel [2005] and Broadie and Kaya [2006]. Kahl and Jackel [2005] propose an application of an implicit Milstein scheme for the square-root diffusion of the variance process, coupled with a particular discretization for the asset process. A completely bias-free scheme was developed by Broadie and Kaya [2006]. However these more complicated schemes have some practical drawbacks, including complexity and lack of computational speed. Andersen [2007] uses the concept of moment matching for the approximation of the vola process in a Heston model. He ends up with an improved convergence of the whole process and such reducing the complexity of a simulation based evaluation of options. Unfortunately all these approaches only deal with the one-dimensional setting where there is need for a multi-dimensional extension. This is presented here in a setting expandable to the Libor model considered in this work.

A recent revolutionary development is the concept of multilevel Monte Carlo simulation. The multilevel idea, which goes back to Heinrich [2001] in fact, was exploited by Giles [2008] for reducing the complexity of SDE simulation. Virtually, the impact of the bias due to an Euler discretization scheme for example, was eliminated by simulating the problem at different discretization levels. Belomestny and Schoenmakers [2011] applied the multilevel idea to the number of sub-simulations in the computation of dual upper bounds for American options using the method of Andersen and Broadie [2004]. In the same spirit policy iterated lower bounds are constructed in Belomestny, Ladkau and Schoenmakers [2012]. Both approaches are summarized in Belomestny, Ladkau and Schoenmakers [2012a]. The key idea is to construct a telescoping sum of correlated differences of standard estimations at different levels. Doing this in a suitable way yields the same accuracy at lower costs or a more accurate approximation at the same costs compared to the standard estimator.

In this thesis the multilevel idea will be utilized in the following setting. Let $Y = \mathbb{E}[Z]$ be the target problem where the random variable Z has to be numerically approximated by Z_M due to M inner (or sub-)simulations. An increasing M indicates a more accurate approximation of the true random variable Z which we would attain by choosing $M = \infty$.

Let further $Y_M = \mathbb{E}[Z_M]$ and let

$$\widehat{\mathcal{Y}}_{N,M} := \sum_{n=1}^{N} Z_M^{(n)}$$

be the standard Monte Carlo estimator of Y_M due to N outer simulations. I.e., $Z_M^{(n)}$ is the n-th independent realization of Z_M. Now the multilevel estimator for Y has the following form,

$$\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} = \frac{1}{n_0} \sum_{j=1}^{n_0} Z_{m_0}^{(j)} + \sum_{l=1}^{L} \frac{1}{n_l} \sum_{j=1}^{n_l} \left(Z_{m_l}^{(j)} - Z_{m_{l-1}}^{(j)} \right),$$

where we have $1 \le m_0 \le ... \le m_L$ and $n_0 \ge ... \ge n_L \ge 1$ for some set of natural numbers m_i, n_i and L. The summands on the right hand side will be independently simulated, where in the leading term m_0 will be significantly smaller than M in the standard estimator.

A severe restriction in modeling financial markets and pricing of financial derivatives is given by the usual assumption that there exists a unique probabilistic model which is known to the decision maker, for instance an option holder. However, this assumption may not apply due to unreliable estimates or scarce data. We then face the situation of model uncertainty, meaning different probabilistic models may be plausible where each model could lead to very different optimal stopping strategies, thus different option values. Therefore a risk-neutral valuation, given by some expected value of the option, might not be appropriate. There exists a vast literature on theory and applications of optimal stopping and control, going back to Wald [1950] and Snell [1952]. In a general probabilistic setting, a robust approach that has recently gained much attention is provided by convex measures of risk considered by Föllmer and Schied [2002], Frittelli and Gianin [2002] and Heath and Ku [2004] among others. By the representation theorem of convex risk measures, a random future reward, say H, is evaluated according to

$$U(H) = \inf_{\mathbb{Q} \in \mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[H] + c(\mathbb{Q})\}, \tag{0.1}$$

where $\mathcal{Q} = \{\mathbb{Q} | \mathbb{Q} \sim \mathbb{P}\}$ is the set of probabilistic models $\mathbb{Q}$ that share the same null sets with a base reference model $\mathbb{P}$, with each $\mathbb{Q}$ attaching a different probability law to the future reward H, and c being a penalty function specifying the plausibility of the model $\mathbb{Q}$. In the literature a convex risk measure is usually defined as $-U(H)$ leading however to the same optimization problem. Models $\mathbb{Q}$ that have "low" plausibility are associated with a high penalty, while models that have "high" plausibility yield a low penalty. Thus the model $\mathbb{Q}$ is considered fully implausible if we have $c(\mathbb{Q}) = \infty$. A conservative worst-case approach is considered by taking the infimum over $\mathcal{Q}$. This is also typical in (deterministic) robust optimization. In a dynamic setting, as considered in this work, time-consistent versions of convex measures of risk play an important role. Such measures of risk were e.g. discussed by Riedel [2004]. Time-consistency means that if a payoff A in every scenario at time t is preferred over another payoff B then A should also be preferred before time t. For dynamic versions of evaluations of the form (0.1), time-consistency is equivalent to the dynamic evaluation satisfying a dynamic programming principle.

The theory of convex measures of risk and ambiguity averse preferences is well-established and their use in the area of optimal stopping problems has recently been developing; see, in particular, Riedel [2009], Krätschmer and Schoenmakers [2010], Bayraktar, Karatzas and Yao [2010], Bayraktar and Yao [2011], Cheng and Riedel [2013] and Øksendal, Sulem and Zhang [2013]. However, the development of numerical methods to practically solve robust optimal stopping problems has not gained much attention. Therefore, we develop a method to practically solve the optimal stopping problem under ambiguity in a general continuous-time BSDE setting. A BSDE can be seen as an SDE with a given terminal condition $Y_T = \xi$ rather than an initial condition. More specifically, the problem is to find a pair of adapted processes (Y, Z) such that

$$dY(t) = -f(t, Y(t), Z(t))dt + Z(t)dW(t), \quad Y(T) = \xi. \tag{0.2}$$

is fulfilled a.s.

Chapter 2: A new multi-factor stochastic volatility model with displacement

This Chapter is based on the ideas of Ladkau, Schoenmakers and Zhang [2013]. The Libor and Swap rate modeling framework, introduced in Section 1.3 and 1.4, was originally established by Miltersen, Sandmann and Sondermann [1997], Brace, Gatarek and Musiela [1997], and Jamshidian [1997]. Still it is considered to be the best framework for pricing interest rate derivatives due to its great flexibility. As already mentioned one could incorporate many sources of randomness of different type and connect these with different volatility structures. We will take a closer look to a few of them and mention their strength and drawbacks. Since this is a problem with many practical applications a feasible (in some sense) calibration procedure to a complete set of liquid market quotes (e.g. caps and swaptions for different strikes and different maturities) is called for. However, this remains a delicate problem. The early versions of the Libor model, based on log-normal assumption for the forward rates, were usually driven by a set of Brownian motions equipped with some deterministic volatility structure. These Libor models, termed *market models,* as they will be introduced in Section 1.3, where quite popular because they allow for analytic cap(let) pricing and (approximate) analytic swaption pricing via Black 76 type formulas. The name market model comes from the fact that using Black's formula for the pricing of caps/swaptions is common market practice. Several problems are associated with the use of the Libor market models (LMM) and Swap market models (SMM). One of the problems arises when calibrating to caplets and swaptions at the same time. For an easy calibration procedure one decides for one of the two models to have closed-form prices available. Unfortunately the two models are contradictory, so we cannot hope for closed-form solutions to both kind of options at the same time. Therefore, one needs approximations going along with some error. Another main drawback of these market models is that they cannot match implied volatility “smile/skew” behavior observed in the cap and swap markets. Moreover, these “smile/skew” effects became even more pronounced over the years. For incorporating the “smile/skew” behavior, e. g. a different kind of volatility structure is called for.

One of the first proposals to this problem was given by Andersen and Andreasen [2000] extending the model in a Constant Elasticity of Variance (CEV) setting. Their idea was to make the diffusion coefficient of the discrete forward rate a non-linear function of the forward rate itself. The model reads

$$\begin{aligned} dL_k(t) &= \sigma_k^T(t)\, dW^{(k+1)}(t) \\ \sigma_k(t) &= \phi(L_k(t))\, \lambda_k(t) \end{aligned}$$

where λ is a bounded vector valued deterministic function and $\phi : [0, \infty] \to [0, \infty]$ may be a non-linear function, too. The model is able to produce monotonically in-/decreasing implied volatility skews and is capable of closed form option pricing, allowing for fast calibration. Clearly the monotonicity is a drawback of this model. This feature is not often seen in reality, in fact much more complicated skew behaviors can be observed nowadays in the markets.

Another approach used by Joshi and Rebonato [2001] is to look at displaced diffusions. This model also admits closed-form pricing but the produced skews only show monotonic behavior, too. The left hand side of such displaced dynamics has the form

$$\frac{dL_j}{L_j + \alpha_j}.$$

In Brigo and Mercurio [2001] a local volatility model consistent with a mixture of log-normal transition densities and some variations on this is proposed. One of the problems in this approach is the rather complicated volatility structure necessary for Monte Carlo simulation of the model in some fixed (e.g. terminal) measure, and the limited flexibility for matching too pronounced “smile/skew” market data.

One further line of research on “smile/skew” explaining Libor models concentrates on Libor models driven by compound Poisson processes like in Glasserman and Kou [2003], or even infinite activity Lévy processes as in Eberlein and Özkan [2005]. Particularly, in Belomestny and Schoenmakers [2006] a specifically structured jump driven Libor model is developed that allows for feasible sequential calibration to cap volatility-strike data for a whole system of maturities. Generally speaking, however, Monte Carlo simulation of jump driven Libor models is rather troublesome and expensive due to an unavoidable complicated drift term. Recently, in Papapantoleon et. al. [2011] an improvement is established in this respect, by constructing Lévy approximations to this Libor drift. Some of these models are able to manipulate the slope and curvature of the volatility skew by changing jump intensity and size. A key feature is the ability to produce sharp short term skews.

A more flexible approach is the usage of stochastic volatility. A very prominent approach in this line is the SABR model (cf. Morini and Mercurio [2007], Hagan and Lesniewski [2008]). The model gives very fast and accurate calibration results by using approximation formulas for the true price by heat kernel expansions. The second reason for its widely usage in practice is that the three parameters the model depends on are easy to interpret. Unfortunately the SABR model uses a geometric Brownian motion to model the volatility process. Volatilities observed at the market very often show a mean-reverting behavior which cannot be reproduced by the SABR model. The SABR model seems to be more favorable for equity as for forward rate pricing.

The mean-reverting property of the volatility observed in the interest derivative market can be modeled for example with the help of a CIR process. Heston was the first who gave a closed form solution to call options on equities following a log-normal dynamic perturbed with a square root volatility process in Heston [1993]. Such a two-factor model is driven by the following dynamics.

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{V(t)}S(t)dW_S(t) \\ dV(t) &= \kappa(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW_V(t) \\ d\langle W_S, W_V\rangle(t) &= \rho dt \end{aligned}$$

This result was extended in Carr and Madan [1999] to a wide class of models. Nowadays it is possible to use the model in situations where the underlying asset has a log-affine structure. A nice property for practical applications is the fact that this approach enables the use of the FFT-algorithm, giving a quasi-closed-form solution to the pricing problem.

In terms of interest-derivatives this idea was used in Wu and Zhang [2006]. They model different forward rates as log-normal processes using a single stochastic factor adopted for forward rate volatilities, following square root dynamics. In order to make the Libor version of the Heston model tractable the existence of an analytical moment generating function is required. Unfortunately this does not exist. To overcome this problem they approximate the forward rate and forward swap-rate processes after a change of measure with a Heston type processes. The approximations were done by the nowadays well known Libor freezing. The approximation error is studied extensively and is believed to be acceptable small. The problem with their approach is the usage of only one volatility process. We belief this is not enough to accurately fit a whole Libor panel due to structural breaks observed during crisis times.

Generalizing the idea of Wu and Zhang [2006], Belomestny, Mathew and Schoenmakers [2011] introduced an extension of the market model using a high-dimensional square root volatility process. This multifactor stochastic volatility model weights each Libor component with its own square root type stochastic factor v_k and a deterministic loading factor β_{ik}.

$$\frac{dL_i}{L_i} = ...dt + \sum_{k=i}^{n-1} \beta_{ik}\sqrt{v_k}dW_k, \quad i = 1, ..., n-1 \tag{0.3}$$

A special structuring makes it possible to preserve the local covariance structure of the market model. After the standard freezing of the respective Libors $L_i(t)$ to $L_i(0)$ in the drift dynamics of L_i their approach yields pure affine Libor dynamics where the number of Brownian motions is equal to the number of Libors. This theoretical convincing model has a main drawback, its calibration. The time to calibrate to forwards of a single maturity growth with each additional step. So fitting to a broad panel the calibration procedure takes to long to have relevant importance for practical applications. Using the proposed iterative calibration procedure may also lead to a cumulative cementation of the market structure. One may not be able to fit to data with structural breaks.

The approaches presented so far could be roughly classified in two parts, namely generalized volatility and jump diffusion driven models. In the interest rate markets stochastic volatility, as the best representative of the first group, is believed to dominate the jump diffusion ansatz, hence we will follow the first approach using the ideas of Wu and Zhang [2006], Belomestny, Mathew and Schoenmakers [2011] and Joshi and Rebonato [2001]. In the multifactor stochastic volatility Libor model with displacement proposed here each forward rate will be equipped with its own volatility process following square root dynamics. The advantage of this model is that each volatility process can be maturity-wise calibrated to the corresponding cap(let)-vola-strike panel. A problem with the model arises due to the fact that even after the usual Libor freezing in the drift part of this model the dynamics yield no affine structure. So further affine approximations are called for to then end up with a Fourier based pricing procedure. The approximated system allows for efficient, quasi-analytical calibration to a complete system of cap/swaption market quotes and performs well even in crisis times where for example structural breaks in the vola-strike-maturity panel can be observed. One would not be able to accurately calibrate to such data in the Wu and Zhang [2006] setting using one volatility process only. The approximations are typically a little bit less accurate then those in Wu and Zhang [2006] but we have a better fit quality.

The target of this work is to extend the setting of Wu and Zhang [2006] by studying the processes

$$\frac{dL_i}{L_i} = ...dt + \beta_i \sqrt{v_i} dW, \quad i = 1, ..., n-1. \tag{0.4}$$

Note that by taking $v \equiv v_i$ Wu-Zhang model can be obtained again. This approach is less restrictive as (0.3) as the number of Brownian motions has not to be equal to the number of Libors, allowing for factor reduction. In contrast to the structure (0.3), the danger of cumulative cementation of the model in a backward recursive calibration is abandoned. However, several technical issues have to be resolved. As a main point, even after standard Libor freezing in the drift of the full stochastic differential equation (SDE) corresponding to (0.4), we do not have an affine Libor model as in Wu and Zhang [2006] and Belomestny, Mathew and Schoenmakers [2011] anymore. That is, the Fourier based quasi-analytical approximation for caps doesn't carry over directly. The same complication shows up when one attempts to derive an approximate affine swap model from (0.4) in order to derive quasi-analytical (Fourier based) swaption approximations. As a solution we will nevertheless construct affine Libor approximations to (0.4) and affine swap rate approximations connected with (0.4), that allow for quasi-analytical cap and swaption pricing again. But, the price we have to pay is that these approximations are typically (a bit) less accurate than the ones in the setting of Wu and Zhang [2006] and Belomestny, Mathew and Schoenmakers [2011]. Careful tests reveal that the approximation procedures developed in this paper are accurate enough for our purposes however. The bottom line and justification of our new approach is the following "philosophical" point of view.

A modeling package that contains only moderately accurate procedures for calibrating to liquid market quotes (e.g. accuracy $\sim 1\%$*), but, which is able to achieve an adequate fitting error (e.g.* $\sim 3\%$ *due to the* 1% *off pricing methods) in an efficient way, is highly preferable in comparison to a modeling package that contains very accurate pricing procedures for calibration (e.g.* $\leq 0.2\%$ *accurate), but, which is unable to achieve an adequate fitting error (e.g.* $\sim 10\%$*), despite of the accurate pricing formulas.*

Indeed, the former package achieves implicitly a fitting quality with respect to the "true model" of about 4%, while the latter package remains left at an unsatisfactory fit of $\sim 10.2\%$. For more flexibility in the model we extend the structure of (0.4) by a standard Gaussian part and a displacement factor α similar to that one used in Joshi and Rebonato [2001]. We will derive a structure

$$\frac{dL_i}{L_i + \alpha_i} = ...dt + \beta_i^T \sqrt{v_i} dW + \gamma_i^T d\widehat{W}, \quad i = 1, ..., n-1 \tag{0.5}$$

where W and $\widehat{W}$ are independent standard Brownian motions, β_i and γ_i are deterministic loading factors and α_i are displacement constants for $i = 1, ..., n-1$. From a technical point of view this gives no further difficulties, neither with regard to the approximate pricing formulas nor with regard to the calibration procedure. From a practical point of view this structure offers more flexibility. If this flexibility is not needed one can easily set $\alpha_i \equiv 0$ or $\gamma_i \equiv 0$ or both together.

We consider here Libor defining zero-bonds $(B_i)_{i=1,\dots,n}$ adapted to the filtration $\mathbb{F}$ generated by some d-dimensional Brownian motion $\mathcal{W}$ living on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

$$\begin{aligned}
\frac{dL_i}{L_i} &= -\sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_i^T \Gamma_j dt + \Gamma_i^T d\mathcal{W}^{(n)} \\
&= \Gamma_i^\top \left(-\sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_j dt + d\mathcal{W}^{(n)} \right) \\
&=: \Gamma_i^\top d\mathcal{W}^{(i+1)}, \quad i = 1, ...n-1
\end{aligned} \tag{0.6}$$

$\mathcal{W}^{(n)}$ is an equivalent Brownian motion under the terminal measure P_n. Since due to the definition of Libors presented in Chapter 1 L_i is a martingale under P_{i+1}, it automatically follows that $\mathcal{W}^{(i+1)}$ in (0.6) is a standard Brownian motion under the equivalent measure P_{i+1}. Finally we note that in the case where the Γ_j are *deterministic* we have the well documented Libor Market Model (LMM) (see for example Brigo and Mercurio [2001] and Schoenmakers [2005] and the references therein).

Chapter 3: Simulation and related optimization ideas

As the topic presented in Chapter 2 is of practical interest one is concerned about the computational complexity of the whole procedure, in particular the performance of the model in a Monte Carlo simulation . For example, a factor reduction, hence a reduction of the driving Brownian motions, provides a first step to accelerate the simulation procedure. The simulation of the Heston type model is an issue to be treated on its own. Taking a simple Euler scheme one has to notice that it will generally fail since discretized Heston volatility may go negative. Several ways out are known, one of them is to adjust the scheme in a "full truncation" sense. We will show later on what is meant with that. Adapting the Euler scheme this way we are still left with poor convergence properties going along with an unfavorable complexity. At least, however, the truncated Euler scheme may be used as a benchmark. Some recent approaches to an efficient discretization of the continuous-time Heston dynamics for purposes of Monte Carlo simulation where made by Kahl and Jackel [2005] and Broadie and Kaya [2006]. Kahl and Jackel [2005] propose an application of an implicit Milstein scheme for the square-root diffusion of the variance process, coupled with a particular discretization for the asset process. Broadie and Kaya [2006] developed a completely bias-free scheme. However these more complicated schemes have some practical drawbacks, including complexity and lack of computational speed.

In order to improve the performance of the multi-dimensional Heston model simulation, we propose a multidimensional extension of the Andersen scheme (cf. Andersen [2007]). As a generalization of Andersen's approach we consider several new algorithms for time-discretization and Monte Carlo simulation of Heston-type stochastic volatility models. The algorithms are based on a careful analysis of the properties of affine stochastic volatility diffusions. Generally speaking, the Andersen method uses the concept of moment matching for the approximation of the vola process in the Heston model. This results in a vast speed up of the whole simulation process. In the numerical tests we consider the model under realistic, i.e. often challenging parameter settings in the sense that the Feller condition is not satisfied.

Chapter 4: Multilevel simulation based policy iteration for optimal stopping – convergence and complexity

This Chapter is based on Belomestny, Ladkau and Schoenmakers [2012]. So far we considered problems of modeling, calibration and the efficient pricing of plain vanilla options in this work. This Chapter is dedicated to a third cornerstone in financial mathematics, the pricing of financial derivatives beyond plain vanilla options. As this is a wide field we here only deal with a particular problem, namely the pricing of high-dimensional American derivatives. Mathematically it turns out that we have to solve stopping problems. A solution to such problems in an efficient way has been a challenge for decades. For low or moderate dimensions, deterministic (PDE) based methods may be applicable, but for higher dimensions Monte Carlo based methods are practically the only way out.

Monte Carlo methods are popular for mainly two reasons, their dimension independent convergence rates and their generic applicability. In the late nineties several regression methods for constructing “good” exercise policies yielding lower bounds for the optimal value were introduced in the financial literature (see Carriere [1996], Longstaff and Schwartz [2001], and Tsistsiklis and Van Roy [2000], for an overview see also Glasserman [2004]). In the past literature, various Monte Carlo algorithms for pricing American options are developed. Many of these approaches are related to the so called backward dynamic programming approach which comes down to a recursive representation of the Snell envelope. Among these methods we mention the random tree method of Broadie, Glasserman and Ha [2004] and the stochastic mesh method of Broadie and Glasserman [2004], the cross-sectional least squares algorithm by Longstaff and Schwartz [2001], and the quantization algorithm by Bally and Pages [2003]. As an alternative to backward dynamic programming, one may search for a suitable parametric family of exercise boundaries and then maximize the solutions of the corresponding family of boundary value problems over the parameters. In Andersen [2000] this concept has been applied in the context of Bermudan swaptions in a LIBOR market model. Bermudan options are in fact American options with a discrete set of exercise dates. Among many other approaches we mention that Kolodko and Schoenmakers [2006] considered a class of policy iterations. In Bender, Kolodko and Schoenmakers [2008] it is demonstrated that the latter approach can be effectively combined with the Longstaff-Schwartz approach.

The basic concept of policy iteration dates back to Howard [1960] (see also Puterman [1994]), where the idea is the following. Given an input stopping strategy one may ask whether this particular strategy is optimal and if not how to improve it. Policy iteration partially answers these questions by providing an improvement step and a maximum number of those steps until the optimum is attained. A detailed probabilistic treatment of a class of policy iterations (that includes Howard’s one as a special case) as well as the description of the corresponding Monte Carlo algorithms is provided in Kolodko and Schoenmakers [2006]. The basic concept of policy iteration will be partially recapped in Section 4.1 providing a Monte Carlo based solution in Section 4.2.

The central result in Kolodko and Schoenmakers [2006] is an iterative construction of the Bermudan Snell envelope via a sequence of stopping times which increases to the (first) optimal stopping time. Using an appropriate window parameter as defined in Bender, Kolodko and Schoenmakers [2008] one can obtain the last optimal stopping time, too. The stopping times coincide if there exists only one optimal stopping time at a given time point. In each iteration step a whole family of stopping times (τ_i) is improved, where i runs through the set of exercise dates, and τ_i is the stopping time for the Bermudan option which is not exercised before date i. In fact, the proposed improvement is inspired by a canonical exercise policy for Bermudan options which is already not far from being optimal usually; namely exercise as soon as the cash-flow dominates all the Europeans ahead. The thus obtained sequence of stopping families naturally induces an increasing sequence of lower approximations of the Snell envelope. This sequence even coincides with the Snell envelope after finitely many steps. However, the main issue is that after each iteration step one obtains an improved approximation of the Snell envelope which ranges over all exercise dates. This is in contrast to the backward dynamic program which requires, for obtaining a value for the Snell envelope at the initial date, a number of steps equal to the total number of exercise dates.

The methods mentioned above commonly provide a (generally suboptimal) exercise policy, hence a lower bound for the optimal value (or for the price of an American product). As a next breakthrough in Monte Carlo simulation of optimal stopping problems in financial context, a dual approach was developed by Rogers [2002] and independently by Haugh and Kogan [2004], related to earlier ideas in Davis and Karatzas [1994]. Due to the dual formulation one considers “good” martingales rather than “good” stopping times. In fact, based on a “good” martingale the optimal value can be bounded from above by an expected path-wise maximum due to this martingale. Probably one of the most popular numerical methods for computing dual upper bounds is the method of Andersen and Broadie [2004]. However, this method has a drawback, namely a high computational complexity due to the need of nested Monte Carlo simulations. In a recent paper, Belomestny and Schoenmakers [2011] mend this problem by considering a multilevel version of the Andersen and Broadie [2004] algorithm.

In this Chapter we consider a new multilevel primal approach due to Monte Carlo based policy iteration. In the spirit of Belomestny and Schoenmakers [2011] (see also Belomestny, Dickmann and Nagapetyan [2013] and Bujok, Hambly and Reisinger [2012]) we here develop a multilevel estimator, where the multilevel concept is applied to the number of inner Monte Carlo simulations needed to construct a new policy, rather than the discretization step size of a particular SDE as in Giles [2008]. In this context we give a detailed analysis of the bias rates and the related variance rates that are crucial for the performance of the multilevel algorithm. In particular, as one main result, we provide conditions under which the bias of the estimator due to a simulation based policy improvement is of order $1/M$ with M being the number of inner simulations needed to construct the improved policy (Theorem 49). The proof of Theorem 49 is rather involved and has some flavor of large deviation theory. The amount of work (complexity) needed to compute, in the standard way, a policy improvement by simulation with accuracy ϵ is equal to $O(\epsilon^{-2-1/\gamma})$ with γ determining the bias convergence rate. As a result, the multilevel version of the algorithm will reduce the complexity by a factor of order $\epsilon^{1/(2\gamma)}$. In this paper we restrict ourself to the case of Howard’s policy iteration (improvement) for transparency, but, with no doubt the results carry over to the more refined policy iteration procedure in Kolodko and Schoenmakers [2006] as well.

Chapter 5: Robust Optimal Stopping

This Chapter is based on Krätschmer, Ladkau, Laeven, Schoenmakers and Stadje [2014]. So far we considered in this work problems where we assumed one particular probabilistic model. In practice there might not exist a unique “true” model. Even if it would exist information about it could be only attained at some cost which might go to infinity for the case of full information. Thus we here introduce a method to solve optimal stopping and control problems under ambiguity. The theory of optimal stopping and control has evolved into one of the most important branches of modern probability and optimization and has a wide variety of applications in many areas, perhaps most notably in operations management, economics and statistics, and finance. There exists a vast literature on both theory and applications of optimal stopping and control, going back to Wald [1950] and Snell [1952], and we mention here only some works related to our setting: Brennan and Schwartz [1985], McDonald and Siegel [1986], Pindyck [1986], Barone-Adesi and Whaley [1987], Dixit [1989], Dixit and Pindyck [1996], Karatzas and Shreve [1998], Dayanik and Karatzas [2003], Guo and Pham [2005], Dasci and Laporte [2005], Peskir and Shiryaev [2006], Øksendal and Sulem [2007], Henderson and Hobson [2013], and Dharma Kwon [2010]. Prime applications are a manufacturer's market entry decision where from the entrance time onwards, the firm will encounter fixed irreversible costs but will at the same time start generating an (uncertain) reward. The goal of the management would be to maximize their present value. Further we have ageing plant closing decision in operations management; a real estate agent's decision to accept a bid or search problems in economics; and the valuation of American-style derivatives in finance. The buyer of such a derivative wants to find the optimal time to exercise the option such that the reward be maximized. These applications naturally lead to an optimal stopping problem.

Since the (future) reward (sequence) is typically uncertain in these applications, it needs to be evaluated using probabilistic methods, and the main target in the above-mentioned papers on standard (classical) theory of optimal stopping is the maximization of the expected reward over a family of stopping strategies. That is, the central object is the expectation of the reward induced by the problem's payoff process. Such a setting requires that the reward's expectation can be unambiguously determined by the decision-maker, which is the case in particular if the reward's probability law under the probability measure of interest is given to the decision-maker. In reality, however, this is quite a restrictive requirement: and it is, in fact, also one of the main criticisms against using a probabilistic approach at all. In many situations the decision-maker faces uncertainty about the true probabilistic model, meaning that the probability law generating the future reward is (partially) unknown and cannot properly be estimated. This is, for instance, the case if estimation is unreliable, data are scarce, or if the evaluation necessarily relies on extrapolating past trends, but past patterns are no longer representative for their future counterpart. Furthermore, in financial decision-making (as in the case of American-style derivatives), investors may need to cope with markets that are inherently incomplete, meaning, in particular, that no unique probabilistic pricing operator exists. In these situations, different probabilistic models may be plausible, each of them potentially leading to very different optimal stopping strategies. Such model uncertainty is usually referred to as ambiguity. In decision theory, the more specific term of Knightian uncertainty (after Knight [1921]) is also employed, to distinguish from decision under uncertainty problems in which the probabilistic model is completely given — the specific case of decision under risk. Approaches that explicitly take ambiguity into account are often referred to as robust approaches.

In a general probabilistic setting, a robust approach that has recently gained much attention is provided by convex measures of risk (Föllmer and Schied [2002], Frittelli and Gianin [2002], and Heath and Ku [2004], extending Artzner, Delbaen, Eber and Heath [1999]; see also the early Ben-Tal [1985] and Ben-Tal and Teboulle [1987]). For applications of convex risk measures in the context of decision and optimization, see e.g., Ruszczyński and Shapiro [2006], Lesnevski, Nelson and Staum [2007], Ben-Tal, Bertsimas and Brown [2010], Choi, Ruszczyński and Zhao [2011], Goovaerts, Kaas and Laeven [2011], Tekaya and da Costa [20013], Laeven and Stadje [2013] and Laeven and Stadje [2013a]. By the representation theorem of convex risk measures, a random future reward is evaluated according to 0.1 with some penalty function c.

A canonical class of penalty functions is provided by ϕ-divergences; see e.g., Ben-Tal and Teboulle [1987]. In this case, the decision-maker starts with a reference model $\mathbb{P}$, which is an approximation or "an educated guess" to the probabilistic model driving the reward H rather than the true model. The decision-maker therefore does not solely rely on the model $\mathbb{P}$ but considers instead a collection of models $\mathbb{Q}$, with esteemed plausibility (or trust) decreasing with their ϕ-divergence measure with respect to the approximation $\mathbb{P}$. A similar approach was adopted by Hansen and Sargent [2001] and Hansen and Sargent [2007] in macroeconomics, using the specific Kullback-Leibler (ϕ-)divergence (or relative entropy; see also Csiszár [1975] and Ben-Tal [1985]). Another special case of interest is given by penalty functions of the form

$$c\,(\mathbb{Q}) = \begin{cases} 0, & \text{if } \mathbb{Q} \in M \subset \mathcal{Q}; \\ \infty, & \text{otherwise}; \end{cases} \tag{0.7}$$

for a fixed set of probabilistic models $M \subset \mathcal{Q}$. In an ambiguity setting the decision-maker here usually starts with a reference measure $\mathbb{P}$ and takes then a worst case approach over all measures 'close' to $\mathbb{P}$. The subclass of penalty functions given by an indicator function as in (0.7) yields evaluations of the form.

$$U(H) = \inf_{\mathbb{Q} \in M} \mathbb{E}_{\mathbb{Q}}\,[H]\,, \tag{0.8}$$

which attaches the same plausibility to all probabilistic models in M; see e.g., Föllmer and Schied [2004] for further details. Note that in this case U corresponds to a coherent risk measure given by $-U(H)$. In a dynamic setting, such as considered in this Chapter, time-consistent versions of convex measures of risk were discussed by Riedel [2004] and have also been considered more recently in e.g., Ruszczyński and Shapiro [2006], Cheridito, Delbaen and Kupper [2006], Ruszczyński [2010], Philpott, de Matos and Finardi [2013], and Laeven and Stadje [2013a]; see also Duffie and Epstein [1992], Chen and Epstein [2002], Shapiro, Dentcheva and Ruszczynski [2009], Chapter 6, and Glasserman and Xu [2013]. For dynamic versions of evaluations of the form (0.1), time-consistency is equivalent to the dynamic evaluation satisfying a dynamic programming principle.

The topic of decision-making under ambiguity, with probabilities of events unknown to the decision-maker, has been extensively studied in economics since the seminal work of Ellsberg [1961]. It has been noted that incorporating ambiguity may not only be of theoretical and normative interest, but can also play a potential role in explaining empirically important failures of a purely risk-based framework (Chen and Epstein [2002]). Popular approaches to decision-making under ambiguity are provided by the multiple priors preferences of Gilboa and Schmeidler [1989] (see also Schmeidler [1989]), also referred to as maxmin expected utility, and the significant generalization of variational preferences developed by Maccheroni, Marinacci and Rustichini [2006]. With linear utility, multiple priors essentially reduces to the evaluation (0.8) while variational preferences reduces to (0.1). Such preferences induce aversion to ambiguity (Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio [2011]). A version of multiple priors was also studied by Huber [1981] in robust statistics; see also the early Wald [1950].

The theory of convex measures of risk and ambiguity averse preferences is well-established and their use in optimal stopping problems has recently been developing; see, in particular, Riedel [2009], Krätschmer and Schoenmakers [2010], Bayraktar, Karatzas and Yao [2010], Bayraktar and Yao [2011], Cheng and Riedel [2013] and Øksendal, Sulem and Zhang [2013]. However, the development of numerical methods to practically solve robust optimal stopping problems may currently be considered breaking ground.

In this Chapter, we develop a method to practically solve the optimal stopping problem under ambiguity in a general continuous-time setting, allowing for general time-consistent convex measures of risk, i.e., all time-consistent dynamic counterparts of (0.1), and general (sequences of) rewards. , i.e., all time-consistent functionals of the form (0.1), including (0.8) as a special case (with some additional compactness assumptions). As to the payoff process, we allow for a general jump-diffusion model specification. By using a continuous-time setting, we are able to utilize the machinery of backward stochastic calculus. The key to our method is to expand two duality theories of a different kind. The first kind of duality theory is the martingale duality approach to standard optimal stopping problems, dating back to Rogers [2002], Haugh and Kogan [2004] and Andersen and Broadie [2004] (see also Davis and Karatzas [1994]). We expand their martingale dual representation to encompass general preference functionals beyond plain conditional expectation. The second kind of duality theory explicates the connection between time-consistent convex measures of risk and backward stochastic differential equations (BSDEs), which we expand to apply to our setting. We note that powerful numerical tools are nowadays available for BSDEs.

Our method is then composed of four steps. First, expanding duality theory of the second kind and using backward stochastic calculus, we construct a suitable Doob martingale from the Snell envelope generated by the optimally stopped and robustly evaluated payoff process. Further we construct the generalized continuation value which will be used in step four as an exercise criterion. Using duality theory, we show that the optimal entrance problem under ambiguity may be seen as a standard American-type optimal exercise problem plus an additional drift given by a penalty which is due to the ambiguity aversion and depends on the volatility with respect to the Brownian motion and the volatility with respect to the Poisson jumps. Second, expanding duality theory of the first kind, we employ this martingale to construct an approximated upper bound to the solution of the optimal stopping problem. Third, we introduce the notion of backward-forward simulation to obtain a genuine upper bound to the solution. Fourth we calculate a lower bound by extracting stopping times from the generalized continuation value. We analyze the asymptotic behavior of our method by deriving its convergence properties. To the best of our knowledge, we are not aware of other practical solution methods for robust optimal stopping problems in the literature so far. Finally, to illustrate the generality of our approach and the relevance of ambiguity to optimal stopping, we supplement the presentation of our method with a few examples of robust optimal stopping problems, including Kullback-Leibler divergences, worst case scenarios, and good-deal bounds.

1. Modeling and Pricing of Interest Rates under No-Arbitrage Assumption

1.1. Arbitrage-free Pricing

This section is dedicated to the modeling of financial instruments. We establish an arbitrage-free framework and introduce concepts like state-price deflator, self-financing trading strategies and complete markets. These financial instruments will be used for modeling zero coupon bonds and therefore forward interest rates later on.

We will make use of the following spaces while fixing some finite time horizon $T_\infty < \infty$.

Definition 1 *For an n-dimensional stochastic process $X = (X(t))_{0 \le t \le T_\infty}$ we define the space*

$$\mathcal{S}^n := \{X : \quad X \text{ continuous semimartingales on } [0, T_\infty]\}$$

where in short hand notation we write $\mathcal{S}^1 := \mathcal{S}$. We further define

$$\mathcal{S}^n_+ := \{X : \quad X \in \mathcal{S}^n,\, X(t) > 0 \quad \forall\, 0 \le t \le T_\infty\}$$

where once again $\mathcal{S}^1_+ := \mathcal{S}_+$.

We consider further an n-dimensional stochastic process $B = (B(t))_{0 \le t \le T_\infty} \in \mathcal{S}^n$ of tradable assets where each risky asset $B_i \in \mathcal{S}$, $i = 1, ..., n$ is defined as a solution to a certain stochastic differential equation (SDE)

$$dB_i(t) = \mu_i(t, B(t))\, B_i(t)\, dt + \sigma_i^T(t, B(t))\, B_i(t)\, dW(t), \quad B_i(0) > 0. \tag{1.1}$$

Lets assume for technical reasons the $\mathbb{R}^n$-valued drift $\mu^T = (\mu_1, ..., \mu_n)$ and the $\mathbb{R}^{m \times n}$-valued volatility $\sigma = (\sigma_1, ..., \sigma_n)$ process to be $\mathbb{F}$-predictable. Given that $\int_0^{T_\infty} |\mu_i(u)|\, du < \infty$ $\mathbb{P}$-a.s. and σ_i is W-integrable for all i, (1.1) admits a unique strong solution (cf. Appendix A.9). Note that the solution of (1.1) is given by

$$B_i(t) = B_i(0) \exp\left(\int_0^t \sigma_i^T(s, B(s))\, dW(s) \right. \tag{1.2}$$
$$\left. + \int_0^t \left(\mu_i(s, B(s)) - \frac{1}{2} \sigma_i^T(s, B(s))\, \sigma_i(s, B(s)) \right) dt \right).$$

Remark 2 *The last equation holds $\mathbb{P}$-a.s. Let us fix the convention that we suppress this property whenever this is clear due to the context.*

Provided that B_i satisfies the Novikov condition (cf. Appendix A.5) we have that B_i is a martingale.

Definition 3 *A process $B = (B(t))_{0 \leq t \leq T_\infty} \in \mathcal{S}^n$ is called a market B (also referred to as price system B). Such a market is said to be arbitrage-free if there exists an adapted process $H = (H(t))_{0 \leq t \leq T_\infty}$ with respect to $\mathbb{F}$ with $H(0) = 1$ and $H(t) > 0$, $0 \leq t \leq T_\infty$, such that HB_i are martingales for all $i = 1, ..., n$. The process H is referred to as the state-price deflator.*

From the facts that HB_i are martingales and $B_i > 0$ one can show with help of the martingale representation theorem that H can be represented as the solution of the following SDE

$$dH(t) = -r(t, B(t)) H(t) dt + \theta^T(t, B(t)) H(t) dW(t) \tag{1.3}$$

with $r \in \mathbb{R}$ and $\theta \in \mathbb{R}^m$ suitably defined.

Remark 4 *By using the concept of a state-price deflator there is no need to assume the existence of a continuously compounding saving account, a tool which might not exist in reality. Moreover H is in fact a function of B, which is why we write the formal dependence of B for r and θ.*

The rate r represents the virtual rate of interest and θ is called the market price of risk. Let us next introduce trading strategies to the market.

Definition 5 *A $\mathbb{R}^n$-dimensional $\mathbb{F}$-predictable process $\pi = (\pi(t))_{0 \leq t \leq T_\infty}$ which is B-integrable is called a trading strategy in the above market $B \in \mathcal{S}^n$. The corresponding value process is defined as*

$$V^\pi := \sum_{i=1}^{n} \pi_i B_i = \pi^T B.$$

π is called a self-financing trading strategy if the corresponding value process can be written as

$$V^\pi(t) = V^\pi(0) + \int_0^t \pi^T(s) dB(s).$$

With the self-financing trading strategy π there is no cash in- or outflow. One only invests the initial capital $V^\pi(0)$ and all changes can be seen as the sum of changes of the individual portfolio components.

Definition 6 *A trading strategy π is admissible if the process*

$$\left(\int_0^t \pi^T(s) d(H(s) B(s)) \right)_{0 \leq t \leq T_\infty}$$

is a true martingale.

This ensures the existence of an almost sure lower bound for the portfolio wealth, excluding doubling strategies for example.

Remark 7 *More generally it would be enough to assume the local martingale property but for our purpose it is more practicable to assume true martingale property. Several advantages come along with that approach, a few of them are given here. We are able to evaluate contingent claims, defined at a later time, by taking the expectation of their payoffs, further the determination of the replicating SFTS, introduced later on, can be done. Last, the existence and uniqueness of the forward Libor process can be guaranteed given a Libor volatility function of linear growth.*

The condition of arbitrage-freeness corresponds to the efficiency of the market in the following sense.

Theorem 8 *Consider an arbitrage-free market $B \in \mathcal{S}^n$, an admissible trading strategy π and a finite time horizon $T \leq T_\infty$. Let the initial condition be given by $V^\pi(0) = \pi^T(0)\,B(0) = 0$ and let π be such that $V^\pi(t) \geq 0$ $\mathbb{P}$-a.s. for all $0 \leq t \leq T$. Then it holds that the terminal value of such an investment is given by $V^\pi(T) = 0$ $\mathbb{P}$-a.s.*

This means one cannot make profits in this market without taking any risks. Before proving the Theorem let us state the following helpful Lemma.

Lemma 9 *Let π be a self-financing trading strategy in the market $B \in \mathcal{S}^n$, H a positive continuous adapted semimartingale, with π being H-integrable. Then π is also a self-financing trading strategy in the market HB.*

Proof. From the self-financing property of π with respect to the market B one has

$$d\left(\pi^T B\right) = \pi^T dB. \tag{1.4}$$

From the last equation we conclude that π is B- integrable. The self-financing property (1.4) needs to be fulfilled for π with respect to the market HB, too. Using Ito's product rule for semimartingales component-by-component twice and (1.4) we have

$$\begin{aligned} d\left(\pi^T HB\right) &= d\left(H\left(\pi^T B\right)\right) \\ &= Hd\left(\pi^T B\right) + \left(\pi^T B\right)dH + \left\langle H, \pi^T B\right\rangle \\ &= \pi^T\left[HdB + BdH + \langle H, B\rangle\right] = \pi^T d\left(HB\right). \end{aligned}$$

Notice that the right hand side is well defined due to Lemma 2.1 in Jamshidian [2001] and further $\langle H, B\rangle$ is well defined. ■

Remark 10 *Lemma 2.1 in Jamshidian [2001] is given for $B \in \mathcal{S}$, however, the Kunita-Watanabe inequality used for the proof is not restricted to the one-dimensional case. Therefore the extended statement here holds true, too.*

Proof of Theorem (8). Due to the assumption of arbitrage-freeness there exists a state-price deflator H, such that HB_i are martingales for all i. From the assumption $\pi(0)^T B(0) = 0$, the facts that HB is a martingale and π is also a self-financing trading strategy in the market HB it follows that $\mathbb{E}\left[\pi^T(T) H(T) B(T)\right] = 0$. From the strict positivity of H we are able to deduce $\mathbb{E}\left[\pi^T(T) B(T)\right] = 0$. Hence, using assumption $V^\pi(t) \geq 0$, one has $\pi^T(T) B(T) = 0$. ■

From (1.3) we deduce

$$
\begin{aligned}
H(t) = \exp\Bigg(&\int_0^t \left(-r(s, B(s)) - \frac{1}{2}\theta(s, B(s))^T \theta(s, B(s))\right) ds \\
&- \int_0^t \theta(s, B(s))^T dW(s)\Bigg).
\end{aligned}
$$

Together with (1.2) we have

$$
\begin{aligned}
H(t) B_i(t) = B_i(0) \exp\Bigg(&\int_0^t (\sigma_i(s, B(s)) - \theta(s, B(s)))^T dW(s) \\
&+ \int_0^t \Big(\mu_i(s, B(s)) - \frac{1}{2}\sigma_i(s, B(s))^T \sigma_i(s, B(s)) \\
&-r(s, B(s)) - \frac{1}{2}\theta(s, B(s))^T \theta(s, B(s))\Big) dt\Bigg).
\end{aligned}
$$

Since the process HB_i is a martingale we have

$$
\begin{aligned}
-\frac{1}{2}\left\|\sigma_i(s, B(s)) - \theta(s, B(s))\right\|^2 &= \mu_i(s, B(s)) - \frac{1}{2}\sigma_i^T(s, B(s))\sigma_i(s, B(s)) \\
&\quad - r(s, B(s)) - \frac{1}{2}\theta^T(s, B(s))\theta(s, B(s)) \\
\mu_i(s, B(s)) &= r(s, B(s)) + \sigma_i^T(s, B(s))\theta(s, B(s)) \qquad (1.5)
\end{aligned}
$$

and we have $d(HB_i) = \int_0^t (\sigma_i - \theta)^T dW$. This can be interpreted in two ways. If the market is arbitrage-free there exists a process H of the form (1.3) such that (1.5) is fulfilled for r and θ. Conversely if for a market B there exist r and θ such that (1.5) is fulfilled and the solution H of (1.3) is such that HB_i satisfies certain integrability conditions, such that HB_i is a martingale, then the market is arbitrage-free.

Definition 11 *A price system $B \in \mathcal{S}^n$ is called complete if state-price deflator H is unique.*

The last definition means that for any investment there exists at least one trading strategy replicating the future terminal wealth of such an investment. With that definition in hand one should ask for situations when a solution to (1.5) exists, hence when there exists an arbitrage-free price system and under which conditions this solution is unique, hence the market is complete. The two questions are partially answered in the following Proposition.

Proposition 12 *Assume that the matrix $\sigma \in \mathbb{R}^{n\times m}$ has full rank m, hence $m \leq n$, for all $(t,\omega) \in [0,T_\infty] \times \Omega$. With $\mathbf{1} := (1,...,1)^T \in \mathbb{R}^n$ we have the following properties for the market.*

- *For $m = n$ the market is arbitrage-free and incomplete.*
- *For $m = n-1$ one has*
    - $\mathbf{1} \notin span\left\{\sigma^T\right\}$*, the market is arbitrage-free and complete*
    - $\mathbf{1} \in span\left\{\sigma^T\right\}$
        - $\mu \in span\left\{\mathbf{1},\sigma^T\right\}$*, the market is arbitrage-free and incomplete*
        - $\mu \notin span\left\{\mathbf{1},\sigma^T\right\}$*, there exist arbitrage possibilities.*
- *For $m < n-1$ one has*
    - $\mu \in span\left\{\mathbf{1},\sigma^T\right\}$*, the market is arbitrage-free and*
        - $\mathbf{1} \notin span\left\{\sigma^T\right\}$*, complete*
        - $\mathbf{1} \in span\left\{\sigma^T\right\}$*, incomplete*
    - $\mu \notin span\left\{\mathbf{1},\sigma^T\right\}$*, there is arbitrage.*

Before giving the proof let us state the following helpful definition.

Definition 13 *Given an $n\times m$-matrix A and an n-dimensional vector b one defines the augmented matrix $(A|b)$ as*

$$(A|b) := \begin{pmatrix} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{pmatrix}.$$

Proof of Proposition (12). The proof is based on three properties well known from linear algebra.

i) $rg\,(A) = rg\,(A|b) \Leftrightarrow b \in span\,\{A\}$

ii) there exists a solution to $Ax = b$ if $rg\,(A) = rg\,(A|b)$

iii) the solution is unique if the columns of A are linearly independent ($rg\,(A) = m$)

Those results can be found Eisenreich [1974] for example.

Rewriting (1.5) as

$$\left(\mathbf{1},\sigma^T\right)\begin{pmatrix} r \\ \theta \end{pmatrix} = \mu \tag{1.6}$$

we set $A = \left(\mathbf{1},\sigma^T\right)$, $x = (r,\theta)^T$ and $b = \mu$. In the case $m = n$ it follows together with the full rank assumption to σ

$$n = \min(n, m+1) \geq rg\left(\mathbf{1},\sigma^T\right) \geq rg\left(\sigma^T\right) = rg(\sigma) = m.$$

So the $n \times (m+1)$-matrix $\left(\mathbf{1},\sigma^T\right)$ has full rank n and so does the augmented matrix $\left(\mathbf{1},\sigma^T|\mu\right)$. Therefore there exists a solution. Following the same arguments one gets $rg\left(\mathbf{1},\sigma^T\right) \leq \min(n, m+1) = n < m+1$ so the solution is not unique. For $m = n-1$ we have a solution for $rg\left(\mathbf{1},\sigma^T\right) = rg\left(\mathbf{1},\sigma^T|\mu\right)$, either for $\mathbf{1} \notin span\left\{\sigma^T\right\}$ or $\mathbf{1} \in span\left\{\sigma^T\right\}$ together with $\mu \in span\left\{\mathbf{1},\sigma^T\right\}$. Hence there exist no arbitrage opportunities. For $\mathbf{1} \notin span\left\{\sigma^T\right\}$ one has $rg\left(\mathbf{1},\sigma^T\right) = m+1 = n$ and the solution is unique, hence the market is complete and incomplete otherwise for $\mathbf{1} \in span\left\{\sigma^T\right\}$, so $rg\left(\mathbf{1},\sigma^T\right) = m < n$. For the missing case of arbitrage possibilities with $\mathbf{1} \in span\left\{\sigma^T\right\}$ and $\mu \notin span\left\{\mathbf{1},\sigma^T\right\}$ we have $rg\left(\mathbf{1},\sigma^T\right) \neq rg\left(\mathbf{1},\sigma^T|\mu\right)$, so no solution. For $m < n-1$ one has a solution if $\mu \in span\left\{\mathbf{1},\sigma^T\right\}$, hence no arbitrage. The solution is unique, and the market complete, if $\mathbf{1} \in span\left\{\sigma^T\right\}$. Otherwise, for $\mu \notin span\left\{\mathbf{1},\sigma^T\right\}$, there exist arbitrage due to the fact that there is no solution as stated in the case $m = n-1$. ■

Next we are going to explain how to price claims, meaning $\mathcal{F}_T$-measurable random variables, with respect to the price system B.

Theorem 14 *Given a complete and arbitrage-free market $B \in \mathcal{S}^n$, for every $0 \leq T \leq T_\infty$, and any random variable $C(T) \in \mathcal{F}_T$, there exists a self-financing trading strategy π and an initial investment $V^\pi(0)$ such that*

$$C(T) = V^\pi(T) = V^\pi(0) + \int_0^T \pi^T(t)\, dB(t).$$

The value $V^\pi(0)$, given by $H^{-1}(0)\,\mathbb{E}^{\mathcal{F}_0}[C(T)\,H(T)]$, is then considered to be the fair price of the claim.

Proof. *For further applications the full rank condition from Proposition (12) will always assumed to be fulfilled, so the proof will be done under this assumption. For a fixed claim C and a state-price deflator H, $\mathbb{E}^{\mathcal{F}_t}[C(T)\,H(T)]$ is a martingale for all $0 \leq t \leq T$. Via the martingale representation theorem (cf. Appendix A.3, Theorem 73) this can be* *expressed as*

$$\mathbb{E}^{\mathcal{F}_t}\left[C(T)\,H(T)\right] = \mathbb{E}\left[C(T)\,H(T)\right] + \int\limits_0^t \tilde{Z}^T(s)\,dW(s) \qquad (1.7)$$

$$= \mathbb{E}\left[C(T)\,H(T)\right] + \int\limits_0^t H(s)\,Z^T(s)\,dW(s)$$

where $H > 0$ $\mathbb{P}$-a.s. was used in the second step. $\left(\tilde{Z}(t)\right)_{0\le t\le T}$ is a predictable process. For any trading strategy π, hence a predictable and W-integrable process, one can write

$$\int\limits_0^t \pi(s)^T d\left(H(s)\,B(s)\right) = \int\limits_0^t \sum_{i=1}^n \pi_i(s)\,d\left(H(s)\,B_i(s)\right) \qquad (1.8)$$

$$= \int\limits_0^t \sum_{i=1}^n \pi_i(s)\,H(s)\,B_i(s)\left(\sigma_i(s,B(s)) - \theta(s,B(s))\right)^T dW(s)$$

$$= \int\limits_0^t H(s)\left(\pi(s)\circ B(s)\right)^T\left(\sigma^T(s,B(s)) - \mathbf{1}\theta^T(s,B(s))\right)dW(s)\,.$$

$\pi(s)\circ B(s)$ denotes the Hadamard product (cf. Appendix A.10). In the second step the martingale property of HB was exploited. By the completeness and the full rank assumption one has $\mathbf{1} \notin span\left\{\sigma^T\right\}$ and $rg\left(\sigma^T\right) = m < n$. We can conclude that $\sigma^T - \mathbf{1}\theta^T$ ($n\times m$-matrix) has full rank m, so m linearly independent rows. It follows that the $m\times m$-matrix $\left(\sigma^T - \mathbf{1}\theta^T\right)^T\left(\sigma^T - \mathbf{1}\theta^T\right)$ is invertible. Next we want to combine (1.7) and (1.8), so we are going to characterize the solution set of the system

$$x^T\left(\sigma^T - \mathbf{1}\theta^T\right) = Z^T. \qquad (1.9)$$

(1.9) can be solved by

$$x^T = Z^T\left(\left(\sigma^T - \mathbf{1}\theta^T\right)^T\left(\sigma^T - \mathbf{1}\theta^T\right)\right)^{-1}\left(\sigma^T - \mathbf{1}\theta^T\right)^T.$$

The solution set contains all solutions to the homogenous system, too. This is given by

$$\mathcal{N} := \left\{v\in\mathbb{R}^n :\, v^T\left(\sigma^T - \mathbf{1}\theta^T\right) = 0\right\}.$$

Note that $\mathcal{N} = span^{\perp}\left\{\sigma^T - \mathbf{1}\theta^T\right\} = span^{\perp}\left\{range\left(\sigma^T\right)\right\}$, so $\dim\mathcal{N} = n - rg\left(\sigma^T\right) = n - m > 0$. The solution set can be written as

$$x = \left(\sigma^T - \mathbf{1}\theta^T\right)\left(\left(\sigma^T - \mathbf{1}\theta^T\right)^T\left(\sigma^T - \mathbf{1}\theta^T\right)\right)^{-1} Z + \mathcal{N}.$$

So there exists a predictable process $\eta \in \mathbb{R}^n$, solving (1.9). Furthermore, due to the degree of freedom one can force the solution to satisfy

$$\eta^T(t)\,\mathbf{1} = \xi^{-1}(t)\,\mathbb{E}^{\mathcal{F}_t}\left[C(T)\,H(T)\right].$$

We therefore have with $v \in \mathcal{N}$

$$\begin{aligned}\eta^T(t)\,\mathbf{1} &= Z^T\left(\left(\sigma^T - \mathbf{1}\theta^T\right)^T\left(\sigma^T - \mathbf{1}\theta^T\right)\right)^{-1}\left(\sigma^T - \mathbf{1}\theta^T\right)^T\mathbf{1} + v^T\mathbf{1}\\ v^T\mathbf{1} &= \xi^{-1}(t)\,\mathbb{E}^{\mathcal{F}_t}\left[C(T)\,H(T)\right]\\ &\quad - Z^T\left(\left(\sigma^T - \mathbf{1}\theta^T\right)^T\left(\sigma^T - \mathbf{1}\theta^T\right)\right)^{-1}\left(\sigma^T - \mathbf{1}\theta^T\right)^T\mathbf{1}.\end{aligned}$$

Choosing v can be interpreted as an additional equation to the system of linear equations, with an augmented matrix $\left(\sigma^T - \mathbf{1}\theta^T, \mathbf{1}\right)$ with $rg\left(\sigma^T - \mathbf{1}\theta^T, \mathbf{1}\right) = rg\left(\sigma^T, \mathbf{1}\right) = m+1$, since $\mathbf{1} \notin span\left\{\sigma^T\right\}$. The nullspace is then given by

$$\mathcal{N}_0 := \left\{v \in \mathbb{R}^n : \, v^T\left(\sigma^T - \mathbf{1}\theta^T\right) \wedge v^T\mathbf{1} = 0\right\}$$

with $\dim \mathcal{N}_0 = n - rg\left(\sigma^T, \mathbf{1}\right) = n - (m-1)$. So for the special case of $m = n-1$ the solution is unique. From (1.7) and (1.8) together with (1.9) one has

$$\begin{aligned}\int_0^t \pi(s)^T\,d\left(H(s)\,B(s)\right) &= \mathbb{E}^{\mathcal{F}_t}\left[C(T)\,H(T)\right] - \mathbb{E}\left[C(T)\,H(T)\right]\\ &= \eta^T(t)\,\mathbf{1}\xi(t) - \eta^T(0)\,\mathbf{1}\xi(0)\\ &= H(t)\,\pi^T(t)\,B(t) - H(0)\,\pi^T(0)\,B(0)\\ &= \pi^T(t)\,H(t)\,B(t) - \pi^T(0)\,H(0)\,B(0),\end{aligned}$$

hence π is a self-financing trading strategy for the price system HB. Applying Lemma 9 π is also a self-financing trading strategy in $H^{-1}(HB)$, so

$$\begin{aligned}\pi^T(t)\,B(t) &= \pi^T(0)\,B(0) + \int_0^t \pi(s)^T\,dB(s)\\ &= H^{-1}(0)\,\mathbb{E}\left[C(T)\,H(T)\right] + \int_0^t \pi(s)^T\,dB(s).\end{aligned}$$

In particular one has

$$C(T) = V^\pi(T) = \pi^T(T)\,B(T)$$

and

$$V^\pi(0) = H^{-1}(0)\,\mathbb{E}\left[C(T)\,H(T)\right].$$

■

Next we want to consider the price of a financial derivative C. We know that in a complete market this derivative is replicable. In the situation of incomplete markets this no longer holds true but the set of replicable claims will be non-empty. For further examination let us assume that C_T is replicable in an arbitrage-free price system B with the self-financing trading strategy π.

$$C(T) = \pi^T(T)\, B(T) = \pi^T(0)\, B(0) + \int_0^T \pi^T(t)\, dB(t)$$

Since the market is arbitrage-free there exists a price deflator H such that HB_i are martingales. From Lemma 9 we know that π is also a self-financing trading strategy in the system HB.

$$H(T)\, C(T) = H(T)\, \pi^T(T)\, B(T) = H(0)\, \pi^T(0)\, B(0) + \int_0^T \pi^T(t)\, d\left(H(t)\, B(t)\right)$$

By the martingale property of the process $\left(\int_0^T \pi^T(t)\, d\left(H(t)\, B(t)\right)\right)_{0\leq t\leq T}$ we conclude by taking the conditional expectation of the last equation

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_t}\left[H(T)\, C(T)\right] &= H(t)\, \pi^T(t)\, B(t) \\
&= H(0)\, \pi^T(0)\, B(0) + \int_0^t \pi^T(s)\, d\left(H(s)\, B(s)\right).
\end{aligned}$$

Hence we have

$$\pi^T(t)\, B(t) = H^{-1}(t)\, \mathbb{E}^{\mathcal{F}_t}\left[H(T)\, C(T)\right]. \tag{1.10}$$

In (1.10) the trading strategy may be not unique. But another trading strategy $\tilde{\pi}$ would lead to the same value of the left-hand side, $\tilde{\pi}^T(t)\, B(t)$. In an incomplete market the price-deflator may also not be unique. However the left-hand side is independent of H, due to the independence of the right-hand side from any particular trading strategy. So this fact will cause no problems, too. So

$$C(t) := H^{-1}(t)\, \mathbb{E}^{\mathcal{F}_t}\left[H(T)\, C(T)\right] \tag{1.11}$$

is a convenient choice for the time t value of the claim C. From the last expression we can also conclude that HC is a martingale.

Let us next describe mathematically what we understand by an instantaneous saving account whose existence is not necessary for the approach presented here. Nevertheless for some parts of this work its existence is assumed. Such an account X with respect to $B \in \mathcal{S}^n$ is characterized by the SDE $dX = \tilde{r}X dt$ for some process $\tilde{r}$, yielding a secure return of $\exp\left(\int \tilde{r}(t)\, dt\right)$ for such an investment where the integral boundaries have to be chosen with respect to the time of investment. We further need X to be a replicable security, thus there exists a nonzero π such that $X = \pi B$. Let us assume for that π that $\pi^T \mathbf{1} = 1$, what can always be obtained for a nonzero strategy after rescaling. From the SDE representation we conclude that X has finite variation so the price of X satisfies $\langle X \rangle = 0$. Further we need $\pi(t) \in \ker\left(\sigma^T(t)\right) \forall\ 0 \leq t \leq T$. The price of X is given by

$$\pi^T(t) B(t) = \pi^T(0) B(0) \exp\left(\int_0^t \pi^T(s) \sigma^T(s, B(s))\, dW(s) \right.$$
$$\left. + \int_0^t \left(\pi^T(s) \mu^T(s, B(s)) - \frac{1}{2} \left\| \pi^T(s) \sigma^T(s, B(s)) \right\|^2 \right) ds \right).$$

With $\pi^T(s) \sigma^T(s, B(s)) = 0$ and (1.5) in vector form, meaning $\pi^T(t) \mu(t, B(t)) = r(t, B(t))$, the dynamics of the saving account are given by

$$X(t) = X(0) \exp\left(\int_0^t r(s, B(s))\, ds\right).$$

A detailed examination of the existence/ non-existence of an instantaneous saving account can be found in Reiß et. al. [2007].

1.2. Measure Change

In the examination done so far all prices were given in the so called real world measure. Next we are going to introduce the concept of change of numéraire or change of measure.

In financial mathematics prices are considered relative to a numéraire, e. g. a currency or gold. This means one measures expectations with respect to some natural quantity. For example the relation between a currency and gold was given by the gold standard in nearly all countries until the beginning of the 20th century. So the currency issued by the national central bank could be exchanged for gold at a certain exchange rate set by the central bank and so transformed to another currency by exchanging with another national central bank. So simply speaking one could measure prices for goods in terms of gold, which can be interpreted as a real world measure, but pays with a currency, somehow related to the quantity of gold. Before stating the concept let us look at the following motivating example. Assume you are investing in a foreign market and the price of your portfolio, claimed in the foreign currency moves upwards. If at the same time the foreign currency loses value against your home one it is possible that your portfolio value seen from your home currency has decreased. So it would be desirable to express prices relative to a certain numéraire, e.g. your home currency. A numéraire can be given by a positive valued asset.

Simply speaking we are looking for a process $\frac{d\mathbb{P}}{d\mathbb{P}_A}$ that makes

$$C(t) = H^{-1}(t)\,\mathbb{E}^{\mathcal{F}_t}[H(T)\,C(T)] = H^{-1}(t)\,\mathbb{E}_A^{\mathcal{F}_t}\left[H(T)\,C(T)\,\frac{d\mathbb{P}}{d\mathbb{P}_A}\right] \tag{1.12}$$

a true statement where $\mathbb{E}_A$ is the expectation with respect to a numéraire A. We are interested in the form of $\frac{d\mathbb{P}}{d\mathbb{P}_A}$ and conditions to $\frac{d\mathbb{P}}{d\mathbb{P}_A}$ and the numéraire A under which (1.12) holds true. These questions are answered in the famous Girsanov theorem, stated without proof.

Theorem 15 *Let the system (1.1) be an arbitrage-free market with price deflator H. Let further A be a positive adapted process such that HA is a martingale. The related probability measure $\mathbb{P}_A$, called the A-numéraire measure is then defined by the Radon-Nikodym derivative*

$$\frac{d\mathbb{P}_A}{d\mathbb{P}} := \frac{A(T_\infty)\,H(T_\infty)}{A(0)\,H(0)}.$$

For any adapted process X it then holds

$$\mathbb{E}_A[X] = \int X d\mathbb{P}_A = \int X \frac{d\mathbb{P}_A}{d\mathbb{P}} d\mathbb{P} = \mathbb{E}\left[X \frac{A(T_\infty)\,H(T_\infty)}{A(0)}\right].$$

If further X is $\mathcal{F}_t$-measurable one has

$$\begin{aligned}
\mathbb{E}_A[X(t)] &= \mathbb{E}\left[\mathbb{E}^{\mathcal{F}_t}\left[X(t)\,\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}^{\mathcal{F}_t}\left[X(t)\,\frac{A(T_\infty)\,H(T_\infty)}{A(0)}\right]\right] \\
&= \mathbb{E}[X(t)\,R(t)]
\end{aligned}$$

because of the martingale property of HA with

$$R(t) := \mathbb{E}^{\mathcal{F}_t}\left[\frac{d\mathbb{P}_A}{d\mathbb{P}}\right] = \frac{A(t)\,H(t)}{A(0)}.$$

The Radon-Nikodym process R is a martingale. More generally we have

$$\mathbb{E}_A^{\mathcal{F}_t}[X(T)] = \frac{\mathbb{E}^{\mathcal{F}_t}\left[X(T)\,\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]}{\mathbb{E}^{\mathcal{F}_t}\left[\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]}.$$

A very useful property in our context is given by the following Lemma.

Lemma 16 *If HC and HA are martingales, then C/A is a $\mathbb{P}_A$-martingale.*

Proof. Assuming that A is a suitable numéraire we have $A > 0$, therefore C/A well defined. For $0 \leq t \leq T \leq T_\infty$ it holds

$$\begin{aligned}
\mathbb{E}_A^{\mathcal{F}_t}\left[\frac{C(T)}{A(T)}\right] &= \frac{\mathbb{E}^{\mathcal{F}_t}\left[\frac{C(T)}{A(T)}\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]}{\mathbb{E}^{\mathcal{F}_t}\left[\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]} \\
&= \frac{\mathbb{E}^{\mathcal{F}_t}\left[\mathbb{E}^{\mathcal{F}_T}\left[\frac{C(T)}{A(T)}\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]\right]}{\mathbb{E}^{\mathcal{F}_t}\left[\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]} \\
&= \frac{\mathbb{E}^{\mathcal{F}_t}\left[\frac{C(T)}{A(T)}\frac{A(T)H(T)}{A(0)}\right]}{\mathbb{E}^{\mathcal{F}_t}\left[\frac{d\mathbb{P}_A}{d\mathbb{P}}\right]} \\
&= \frac{\mathbb{E}^{\mathcal{F}_t}\left[\frac{C(T)H(T)}{A(0)}\right]}{\frac{A(t)H(t)}{A(0)}} \\
&= \frac{\frac{C(t)H(t)}{A(0)}}{\frac{A(t)H(t)}{A(0)}} \\
&= \frac{C(t)}{A(t)}.
\end{aligned}$$

■

With that result in hand we are now able to present (1.11) in terms of different measures.

Proposition 17 *Like in Theorem 15 let HA be a martingale, $A > 0$ and further C an $\mathcal{F}_T$-claim that can be hedged by a SFTS. Then the time t-price of a claim C under the measure $\mathbb{P}_A$ is then given by*

$$C(t) = A(t)\,\mathbb{E}_A^{\mathcal{F}_t}\left[\frac{C(T)}{A(T)}\right]$$

Proof. Since C is an $\mathcal{F}_T$-claim HC is a martingale. Together with Lemma 16 it follows that C/A is a $\mathbb{P}_A$-martingale so the statement easily follows. ■

Such a numéraire can be given for example by a bond B_i as defined in (1.1). Therefore we have that $\frac{B_j}{B_i}$ is $\mathbb{P}_{B_i}$-martingale, a fact that will play a central role in further examination.

For reasons of clearness we will suppress the dependences of the coefficients on time where ever it is clear from the context.

1.3. Libor Rate Process and Libor Market Model

Using the results of the previous subsections we are going to model the Libor rate process while having a closer look at compound interest and so in-year payments of zero coupon bonds. Further the pricing of related options, namely cap(let)s in the Libor market model (LMM) is discussed.

Let us now study the arbitrage-free price system (1.1) consisting of zero coupon bonds B_i. Let us therefore consider a fixed tenor structure $\mathcal{T} := \{T_1, ..., T_n\}$ satisfying $0 = T_0 < T_1 < ... < T_n < T_\infty$. Define further the so called day-count fractions $\delta_i := T_{i+1} - T_i$. With respect to this tenor structure the zero coupon bonds processes B_i, $i = 1, ..., n$, live on the interval $[0, T_i]$ and end up with their face value $B_i(T_i) = 1$. Let us interpret these bonds as an interest rate account where an investment in bond B_i over time period $[t, T_i]$ yields

$$\frac{B_i(T_i) - B_i(t)}{B_i(t)} =: r_i(t) \frac{T_i - t}{1(y)}, \quad i = 1, ..., n. \tag{1.13}$$

A rearrangement of (1.13) together with $B_i(T_i) = 1$ yields

$$1 + r_i(t) \frac{T_i - t}{1(y)} = \frac{1}{B_i(t)}, \quad i = 1, ..., n. \tag{1.14}$$

We then have the yield of an investments in the two periods $[t, T_i]$ and $[t, T_{i+1}]$ by investing at time t in B_i and B_{i+1} respectively. The natural question arising is for a connection between these two quantities. So we are looking for a forward rate meaning an arbitrage-free investment over the time period $[T_i, T_{i+1}]$ and its corresponding yield $r_{i,i+1}$ at time t. This investment fulfills

$$(1 + r_i(t) \frac{T_i - t}{1(y)})(1 + r_{i,i+1}(t) \frac{T_{i+1} - T_i}{1(y)}) = 1 + r_{i+1}(t) \frac{T_{i+1} - t}{1(y)},$$

where on the left hand side a reinvestment took place at time T_i. From the last equation we get using (1.14)

$$L_i(t) := r_{i,i+1}(t) = \frac{1(y)}{\delta_i}(\frac{B_i(t)}{B_{i+1}(t)} - 1), \quad i = 1, ..., n-1 \tag{1.15}$$

which forms a system of forward rates. These forward rates, called Libor rates, respect in-year interest payments, that means L_i is the annualized effective forward rate contracted in t for a loan over $[T_i, T_{i+1}]$. In T_{i+1} this pays an interest amount of $\delta_i L_i(T_i)$ on a notional of 1. We here suppress the dependence of L_i on B for reasons of clearness. With (1.1) and Appendix A.7 the Libor dynamics can be reformulated to

$$\begin{aligned}
dL_i &= \frac{1}{\delta_i} d\left(\frac{B_i}{B_{i+1}} - 1\right) \\
&= \frac{1}{\delta_i} d\frac{B_i}{B_{i+1}} \\
&= \frac{1}{\delta_i} \frac{B_i}{B_{i+1}} \left(\left(\sigma_i^T - \sigma_{i+1}^T\right) dW + \left(\mu_i - \mu_{i+1} - \sigma_{i+1}^T (\sigma_i - \sigma_{i+1})\right) dt\right).
\end{aligned}$$

Together with (1.5), namely

$$\mu_i = r + \sigma_i^T \theta,$$

one has

$$dL_i = \frac{1}{\delta_i}\frac{B_i}{B_{i+1}}\left(\left(\left(\sigma_i^T - \sigma_{i+1}^T\right)\theta - \sigma_{i+1}^T\left(\sigma_i - \sigma_{i+1}\right)\right)dt + \left(\sigma_i^T - \sigma_{i+1}^T\right)dW\right) \tag{1.16}$$
$$= \frac{1}{\delta_i}\frac{B_i}{B_{i+1}}\left(\left(\left(\sigma_i^T - \sigma_{i+1}^T\right)\theta - \left(\sigma_i^T - \sigma_{i+1}^T\right)\sigma_{i+1}\right)dt + \left(\sigma_i^T - \sigma_{i+1}^T\right)dW\right)$$
$$= \frac{1}{\delta_i}\frac{B_i}{B_{i+1}}\left(\left(\sigma_i^T - \sigma_{i+1}^T\right)\left(\theta - \sigma_{i+1}\right)dt + \left(\sigma_i^T - \sigma_{i+1}^T\right)dW\right).$$

From (1.15) we obtain an equivalent formulation

$$\frac{B_i(t)}{B_{i+1}(t)} = 1 + \delta_i L_i(t)$$

which allows us to reformulate (1.16).

$$dL_i = \frac{1}{\delta_i}\left(1 + \delta_i L_i\right)\left(\left(\sigma_i^T - \sigma_{i+1}^T\right)\left(\theta - \sigma_{i+1}\right)dt + \left(\sigma_i^T - \sigma_{i+1}^T\right)dW\right) \tag{1.17}$$

To be able to represent (1.17) more compact we introduce the volatility process $\gamma_i = (\gamma_i(t, B(t)))_{0 \le t \le T_i}$ and define implicitly

$$L_i(t)\,\gamma_i^T(t, B(t)) := \frac{1}{\delta_i}\left(1 + \delta_i L_i(t)\right)\left(\sigma_i^T(t, B(t)) - \sigma_{i+1}^T(t, B(t))\right). \tag{1.18}$$

One gets

$$dL_i = L_i\gamma_i^T\left(\theta - \sigma_{i+1}\right)dt + L_i\gamma_i^T dW$$
$$= L_i\gamma_i^T\left(dW + \left(\theta - \sigma_{i+1}\right)dt\right).$$

By defining the drifted Brownian motion

$$dW^{(i+1)}(t) := dW(t) + \left(\theta(t, B(t)) - \sigma_{i+1}(t, B(t))\right)dt$$

we get a representation in the measure $\mathbb{P}_{B_{i+1}}$ related to $W^{(i+1)}$.

$$L_i(t) = \exp\left(\int_0^t \gamma_i^T dW(s) + \int_0^t \left(\gamma_i^T\left(\theta - \sigma_{i+1}\right) - \frac{1}{2}\gamma_i^T\gamma_i\right)ds\right) \tag{1.19}$$
$$= \exp\left(\int_0^t \gamma_i^T\left(dW(s) + \left(\theta - \sigma_{i+1}\right)ds\right) - \frac{1}{2}\int_0^t \gamma_i^T\gamma_i ds\right)$$
$$\Leftrightarrow dL_i = L_i\gamma_i^T dW^{(i+1)} \tag{1.20}$$

(1.20) are the dynamics under the so called forward measure $\mathbb{P}_{B_{i+1}}$. It is clear from (1.15) and Lemma (16) that L_i is a martingale under $\mathbb{P}_{B_{i+1}}$. So from (1.20) and the martingale representation theorem we conclude that $W^{(i+1)}$ is a Brownian motion under the measure $\mathbb{P}_{B_{i+1}}$. This holds true for all $i = 0, ..., n-1$.

Corollary 18 *From (1.20) and the martingale representation theorem we see that*

$$W^{(j)}(t) = W(t) + \int_0^t (\theta(s, B(s)) - \sigma_j(s, B(s)))\, ds, \quad 0 \le t \le T_j \wedge T_{n-1}$$

is an m-dimensional Brownian motion under the measure $\mathbb{P}_{B_j}$, $j = 1, ..., n$.

More generally the representation for L_i under some measure $\mathbb{P}_{B_k}$, $k = 1, ..., n$, is given by

$$\begin{aligned} L_i(t) &= \exp\left(\int_0^t \gamma_i^T (dW(s) + (\theta - \sigma_k)\, ds) \right. \\ &\quad \left. + \int_0^t \left(-\frac{1}{2} \gamma_i^T \gamma_i + \gamma_i^T (\theta - \sigma_{i+1} - (\theta - \sigma_k))\, ds \right) \right) \\ &= \exp\left(\int_0^t \gamma_i^T dW^{(k)}(s) + \int_0^t \left(\gamma_i^T (\sigma_k - \sigma_{i+1}) - \frac{1}{2} \gamma_i^T \gamma_i \right) ds \right). \end{aligned}$$

So

$$\begin{aligned} dL_i &= L_i \gamma_i^T (\sigma_k - \sigma_{i+1}) + L_i \gamma_i^T dW^{(k)} \\ &= L_i \gamma_i^T \left(\sum_{j=k}^{i} (\sigma_j - \sigma_{j+1}) - \sum_{j=i+1}^{k-1} (\sigma_j - \sigma_{j+1}) \right) dt + L_i \gamma_i^T dW^{(k)} \\ &= \sum_{j=k}^{i} \frac{\delta_j L_i L_j}{(1 + \delta_j L_j)} \gamma_i^T \gamma_j dt - \sum_{j=i+1}^{k-1} \frac{\delta_j L_i L_j}{(1 + \delta_j L_j)} \gamma_i^T \gamma_j dt + L_i \gamma_i^T dW^{(k)} \end{aligned}$$

where empty sums are defined to be zero and the last equality follows with (1.18). As a special case we state a widely used measure, the so called terminal measure $\mathbb{P}_{B_n}$ with dynamics

$$dL_i = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_i L_j}{(1 + \delta_j L_j)} \gamma_i^T \gamma_j dt + L_i \gamma_i^T dW^{(n)}.$$

Next we examine the existence of the Libor process, formulating assumptions for the coefficients of the process. A possible set of assumptions is given in Lemma 2.3 of (Jamshidian [2001]).

Lemma 19 *Let $(W(t))_{0 \le t \le T_\infty}$ be an m-dimensional Brownian motion and the scalar functions $\mu_i(t, x)$ as well as the vector valued function $\gamma_i(t, x) \in \mathbb{R}^m$, $0 \le t \le T_\infty$, $x \in \mathbb{R}_+^{n-1}$, $i = 1, ...n-1$ be measurable, bounded and locally Lipschitz continuous in x.*

Then there exists a unique Ito-process $X_i > 0$ with a given initial condition $X(0) \in \mathbb{R}_+^{n-1}$ satisfying the SDE

$$dX_i = X_i \mu_i(t, X)\, dt + X_i \gamma_i^T(t, X)\, dW \quad i = 1, ..., n-1. \tag{1.21}$$

The solution of the last SDE is square integrable and its quadratic variation is integrable.

Proof. The result follows from the standard existence and uniqueness theorem (cf. Appendix A.9) where it is enough to show both properties for the log-transformed SDE

$$dY_i = v_i(t, Y)\, dt + \lambda_i^T(t, Y)\, dW$$

where

$$\lambda_i(t, y) := \gamma_i(t, e^y) \quad v_i(t, y) := \mu_i(t, e^y) - \|\lambda_i(t, y)\|^2 / 2.$$

Obviously v_i and λ_i satisfy the assumptions of Theorem 76, so an upper bound for the second moment is given by

$$\mathbb{E}\left[\|X_i(t)\|^2\right] < K\left(1 + \|X_i(0)\|^2\right) \exp(Kt), \quad 0 \le t \le T_\infty.$$

We further have

$$\mathbb{E}\left[\langle X_i \rangle(t)\right] \le c\mathbb{E}\left[\|X_i(t)\|^2\right], \quad 0 \le t \le T_\infty,$$

for some constant c. ■

Corollary 20 *Let $(W(t))_{0 \le t \le T_\infty}$ and $\gamma_i(t, x)$, $0 \le t \le T_\infty$, $x \in \mathbb{R}_+^{n-1}$, as in Lemma 19. Then there exists a unique Ito-process $X_i > 0$ with a given initial condition $X(0) \in \mathbb{R}_+^{n-1}$ satisfying the SDE*

$$dX_i(t) = -\sum_{j=i+1}^{n-1} \frac{X_i(t) X_j(t) \gamma_i^T(t, X) \gamma_j(t, X)}{1 + X_j(t)} dt + X_i(t) \gamma_i^T(t, X)\, dW(t)$$

for all $i = 1, ..., n-1$. Furthermore the processes defined by

$$Y_i := (1 + X_i) \cdots (1 + X_{n-1}), \quad i = 1, ...n-1,$$

are square integrable martingales.

Proof. From the existence and uniqueness of the solution of (1.21) and the facts that $(X_i/(1 + X_i))$ is bounded and locally Lipschitz as well as the product and the sum of bounded, locally Lipschitz functions we deduce that there exists a unique process $X_i > 0$ satisfying the initial condition $X_i(0) \in \mathbb{R}_+$ solving

$$dX_i = -\sum_{j=i+1}^{n-1} \frac{X_i X_j \gamma_i^T \gamma_j}{1 + X_j} dt + X_i \gamma_i^T dW. \tag{1.22}$$

Furthermore we are going to show that the processes given by

$$Y_i = (1 + X_i) \cdots (1 + X_{n-1}), \quad i = 1, ..., n-1$$

are square integrable martingales. Note that it holds $Y_i = (1 + X_i) Y_{i+1}$. With

$$\lambda_i (t, X) := \sum_{j=i}^{n-1} \frac{X_j \gamma_j (t, X)}{1 + X_j}$$

we are going to show by backward induction

$$dY_i = \lambda_i^T Y_i dW.$$

So Y_i are local martingales. The statements hold true for $i = n-1$ due to

$$dY_{n-1} = dX_{n-1} = X_{n-1}\gamma_{n-1}^T dW = Y_{n-1}\frac{X_{n-1}\gamma_{n-1}^T}{1 + X_{n-1}} dW = Y_{n-1}\lambda_{n-1}^T dW$$

and we have further

$$\begin{aligned}
dY_{i-1} &= dY_i + d(X_{i-1}Y_i) \\
&= dY_i + X_{i-1}dY_i + Y_i dX_{i-1} + \langle X_{i-1}, Y_i \rangle \\
&= (1 + X_{i-1}) \lambda_i^T (t, X) Y_i dW \\
&+ X_{i-1}\gamma_{i-1}^T (t, X) Y_i dW - X_{i-1}\gamma_{i-1}^T (t, X) Y_i \sum_{j=i}^{n-1} \frac{X_j \gamma_j (t, X)}{1 + X_j} dt \\
&+ X_{i-1}\gamma_{i-1}^T (t, X) \lambda_i (t, X) Y_i dt \\
&= (1 + X_{i-1}) \lambda_i^T (t, X) Y_i dW + X_{i-1}\gamma_{i-1}^T (t, X) Y_i dW \\
&= \left((1 + X_{i-1}) \lambda_i^T (t, X) \frac{Y_i}{Y_{i-1}} + X_{i-1}\gamma_{i-1}^T (t, X) \frac{Y_i}{Y_{i-1}} \right) Y_{i-1} dW \\
&= \left(\lambda_i^T (t, X) (t, X) + X_{i-1}\gamma_{i-1}^T (t, X) \frac{1}{1 + X_{i-1}} \right) Y_{i-1} dW \\
&= \lambda_{i-1}^T (t, X) Y_{i-1} dW.
\end{aligned}$$

Since λ_i is a bounded process it follows that Y_i is in fact square integrable by standard arguments as well as $\langle Y_i \rangle$ is integrable. ■

The dynamics of the processes X_i are derived by backward induction. Now replacing X_i by $\delta_i L_i$ and $\gamma_i (t, X)$ by $\gamma_i (t, L)$ in (1.22) yields that

$$\begin{aligned}
dL_i (t) = &- \sum_{j=i+1}^{n-1} \frac{\delta_j L_i (t) L_j (t) \gamma_i^T (t, L(t)) \gamma_j (t, L(t))}{1 + \delta_j L_j (t)} dt \\
&+ L_i (t) \gamma_i^T (t, L(t)) dW(t)
\end{aligned} \tag{1.23}$$

has a unique solution. Furthermore to apply our results obtained so far we need to link the solution of (1.23) with an arbitrage-free price system given a volatility structure γ. This link will be provided in the following Proposition following Theorem 7.1 of Jamshidian [2001]. The process Y is related to the Radon-Nikodym derivative. The martingale property ensures that we are able to change the measure via Girsanov's Theorem.

Proposition 21 *Let $\mathbb{Q}$ be a measure equivalent to $\mathbb{P}$. and $B(0) \in \mathbb{R}_+^n$ be given such that $L(0) \in \mathbb{R}_+^{n-1}$ is defined via (1.15). Further let the process $\gamma(t, L) = (\gamma_i(t, L))_{i=1,\dots,n-1}$, $\gamma_i \in \mathbb{R}^m$, $0 \leq t \leq T_\infty$, $x \in \mathbb{R}_+^{n-1}$ be measurable, bounded and locally Lipschitz continuous in L. Then there exists an arbitrage-free price system $B \in \mathcal{S}_+^n$, starting from $B(0)$ with $B_i(T_i) = 1$ and a price deflator H such that (1.15) and (1.23) are fulfilled and $\mathbb{Q}$ is given by $\mathbb{P}_{B_n}$.*

Proof. Note from Corollary (20) that we have a unique solution L for (1.23) with $Y_i = (1 + \delta_j L_i) \cdots (1 + \delta_{n-1} L_{n-1})$, $i = 1, \dots, n-1$ being square integrable $\mathbb{Q}$-martingales for some measure $\mathbb{Q}$. Let $B_n \in \mathcal{S}_+$ with initial condition $B_n(0) > 0$ such that $B_n(T_n) = 1$ and $B_n(T_i) = 1/Y_i(T_i)$, $i = 1, \dots, n-1$ is given. With $B_i := B_n Y_i$ it holds $B_i(T_i) = 1$. Define further

$$H(t) := B_n(0)\, \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}[d\mathbb{Q}/d\mathbb{P}] / B_n(t). \tag{1.24}$$

From (1.24) and the tower property of the conditional expectation it is clear that HB_n is a $\mathbb{P}$-martingale and moreover HB_i are $\mathbb{P}$-martingales for all i, too. This follows from the fact that $B_i/B_n = Y_i$ are $\mathbb{Q}$-martingales.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}[H(T) B_i(T)] &= B_n(0)\, \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}\left[B_i(T)\, \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_T}[d\mathbb{Q}/d\mathbb{P}] / B_n(T)\right] \\
&= B_n(0)\, \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_T}\left[\frac{B_i(T_\infty)}{B_n(T_\infty)}\right] \\
&= B_n(0)\, \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_T}\left[\frac{B_i(T_\infty)}{B_n(T_\infty)} d\mathbb{Q}/d\mathbb{P}\right] \\
&= B_n(0)\, \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}\left[\frac{B_i(T_\infty)}{B_n(T_\infty)} d\mathbb{Q}/d\mathbb{P}\right] \\
&= B_n(0)\, \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[\frac{B_i(T_\infty)}{B_n(T_\infty)}\right] \\
&= B_n(0)\, \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}[d\mathbb{Q}/d\mathbb{P}]\, B_i(t) / B_n(t) \\
&= H(t)\, B_i(t)
\end{aligned}$$

With H being a price deflator for the system B it follows (cf. Appendix A.4) that the Radon-Nikodym derivative is given by $d\mathbb{Q}/d\mathbb{P} = \frac{H(T_\infty)Q(T_\infty)}{Q(0)}$, where Q denotes the numéraire associated with the measure $\mathbb{Q}$. We then have $\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}[d\mathbb{Q}/d\mathbb{P}] = \frac{H(t)Q(t)}{Q(0)}$ and by comparing with (1.24) and the uniqueness of the Radon-Nikodym process we conclude $Q = B_n$, so $\mathbb{Q} = \mathbb{P}_{B_n}$. ■

To ensure $B_i(T_i) = 1$ we have to let B_n pass through given random variables at time points 0, T_1, ..., T_n. Between these points B_n can be interpolated in any kind of style, so there is no uniqueness of B. However, the Libor process L_i , defined as the solution of (1.23), is independent of the choice of interpolation. We conclude that $B_i(T_j)$ is also independent of the choice of interpolation, because with $B_i(T_i) = 1$ it holds

$$B_i(T_j) = \frac{B_i(T_j)}{B_j(T_j)} = \prod_{k=j}^{i-1} \frac{B_{k+1}(T_j)}{B_k(T_j)} = \prod_{k=j}^{i-1} \frac{1}{1+\delta_k L_k(T_j)}, \quad j < i.$$

Given a bounded locally Lipschitz continuous volatility function $\gamma : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{(n-1)\times m}$ and assuming full rank m, $m \leq n-1$ for $\gamma(t, L)$ we consider the Libor process $L(t)$ as solution of (1.23) and the corresponding arbitrage-free price system B. Again the question of pricing and hedging arises and is partially answered in the following Theorem.

Theorem 22 *Given an arbitrage-free price system (1.1) and the corresponding Libor process (1.15) we consider an n-dimensional Brownian motion $W^{(n)}$. Assume that the volatility process γ has full rank m, where $m < n$, and that it is predictable with respect to the filtration $\mathcal{F}^{(n)}$ generated by $W^{(n)}$. In that case any claim $C(T)$, where $C(T)/B_n(T)$ is measurable with respect to $\mathcal{F}_T^{(n)}$, can be hedged, and therefore priced, by a SFTS π in the bond system B.*

Proof. The proof is similar to that one of Theorem 14. Therefore we will only sketch the proof. From the measurability assumption of $C(T)/B_n(T)$ we deduce that the process is a martingale. Together with the martingale representation Theorem (cf. Appendix A.3, Theorem 73) there exists an $\mathcal{F}^{(n)}$-measurable process Z such that

$$\frac{C(t)}{B_n(t)} = \mathbb{E}_{\mathbb{P}_{B_n}}^{\mathcal{F}_t}\left[\frac{C(T)}{B_n(T)}\right] = \mathbb{E}_{\mathbb{P}_{B_n}}^{\mathcal{F}_0}\left[\frac{C(T)}{B_n(T)}\right] + \int_0^t Z^T(s)\, dW^{(n)}(s).$$

Since the vector-valued process L is adapted with respect to $\mathcal{F}^{(n)}$ we conclude that the system $(B/B_n) := (B_i/B_n)_{i=1,\ldots,n-1}$, is an adapted $\mathbb{P}_{B_n}$-martingale and so there exists a process v such that $d(B/B_n) = v dW^{(n)}$. v is of dimension $n-1 \times m$. Since γ has full rank it is easy to see that v has full rank, too. With $\theta := v\left(v^T v\right)^{-1} Z$ it holds

$$\theta^T d(B/B_n) = Z^T dW^{(n)} = d(C/B_n).$$

Defining $\pi_i := \theta_i$, $i = 1, ..., n-1$ and $\pi_n := C/B_n - \theta^T(B/B_n)$ we get

$$d\left(\left(\pi^T B\right)/B_n\right) = d(C/B_n) = \theta^T d(B/B_n) = \pi^T d(B/B_n).$$

We determine that π is a SFTS in the market B and the price of the claim is given by

$$C(t) = B_n(t)\,\mathbb{E}^{\mathcal{F}_t}_{\mathbb{P}_{B_n}}\left[\frac{C(T)}{B_n(T)}\right] = \pi^T(t)\,B(t) = \pi^T(0)\,B(0) + \int_0^t \pi^T(s)\,dB(s)\,.$$

■

An important financial instrument related to Libors are caps. A cap on a notional of 1 unit over the time period $[T_p, T_q]$ with strike K pays floating spot Libor capped by K on the settlement dates within this period. Its fair present value in $t \leq T_p$ is given by

$$Cap_{p,q,K}(t) := \sum_{j=p}^{q-1} \delta_j B_{j+1}(t)\,\mathbb{E}^{\mathcal{F}_t}_{j+1}\left[(L_j(T_j) - K)^+\right].$$

Hereby, in shorthand notation, E_{i+1} is the expectation with respect to the measure associated to the bond B_{i+1}. Such a cap consists of one-periodic caps, called caplets. Due to the linear dependence it suffices to focus on caplets where we formally define

$$C_j(t) := C_{j,K}(t) := Cap_{j,j+1,K}(t)\,. \tag{1.25}$$

A caplet C_j as a function of L_j is priced in its canonical measure $\mathbb{P}_{B_{j+1}}$via Proposition 17 where L_j is a martingale. As can be seen from the definition (1.25) a caplet is a call option on the Libor with strike K. The equivalent to put options in this context is called a floorlet (respectively floor). The simplest version where we can calculate these options in closed-form is in the case of the so called Libor Market Model (LMM).

Definition 23 *If $t \to \gamma(t)$ is a deterministic process the corresponding Libor model is called a Libor market model (LMM).*

Proposition 24 *For a deterministic volatility structure γ the price of a caplet over period $[T_i, T_{i+1}]$ with strike K is, via the Black-Scholes formula, given by*

$$\delta_i B_{i+1}(t)\,(L_i(t)\mathcal{N}(d_+) - K\mathcal{N}(d_-))$$

where $\mathcal{N}$ is the cumulative distribution function of the normal distribution and

$$d_\pm := \frac{\ln(L_i(t)/K) \pm \left(\sigma_j^B\right)^2 (T_j - t)/2}{\sigma_j^B\sqrt{T_j - t}}, \quad \left(\sigma_j^B\right)^2 := \frac{1}{T_j - t}\int_t^{T_j} \|\gamma(s)\|^2\,ds.$$

Proof. Since γ_i is deterministic L_i is a log-normal martingale under the measure $\mathbb{P}_{B_{i+1}}$. Further we have equality in distribution of

$$\int_t^{T_i} \gamma_i^T(s)\,dW^{(i+1)}(s) = \sigma_j^B\sqrt{T_j - t\xi}$$

for ξ being a standard normal distributed variable. The statement then follows from the Black-Scholes formula and

$$L_i(T_i) \overset{distr.}{=} L_i(t) \exp\left(-\frac{1}{2} \int\limits_t^{T_i} \|\gamma_i(s)\|^2 \, ds + \sigma_j^B \sqrt{T_j - t} \xi \right).$$

■

1.4. Swap Rate Process and Swap Market Model

This Subsection is devoted to the modeling of forward swap rates and the pricing of swaptions, derivatives related to swap rates. The pricing will be done under the assumptions of the swap market model.

We here deal with another standard derivative at the interest market, the swaption. This is an option to enter a swap which we will introduce first. In general in a swap contract one exchanges floating legs against fixed legs on predetermined dates. One distinguishes between payer and receiver swaps. Holding a payer swap contract one receives the floating leg and pays the fixed one. For the receiver swap it works the other way around. We will here focus on the payer swap and the payer swaption related to it. Let the floating leg be given by Libor rates. The time t-value of a swap on a notional of 1 unit with legs from T_p to T_q and a fixed exchange rate K is given by

$$\begin{aligned} Swap_{p,q,K}(t) &:= \sum_{j=p}^{q-1} \delta_j B_{j+1}(t) \left(L_j(t) - K \right) \\ &= B_p(t) - B_q(t) - \sum_{j=p}^{q-1} \delta_j K B_{j+1}(t). \end{aligned} \tag{1.26}$$

The swap rate is the rate $S_{p,q}$ which makes the present value of this contract zero

$$S_{p,q}(t) := \frac{B_p(t) - B_q(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)} =: \frac{B_p(t) - B_q(t)}{B_{p,q}(t)}, \tag{1.27}$$

where $B_{p,q}(t) = \sum_{j=p}^{q-1} \delta_j B_{j+1}(t)$ is called the annuity numéraire. From (1.27) we deduce that $S_{p,q}$ is a martingale under the probability measure $\mathbb{P}_{p,q} := \mathbb{P}_{B_{p,q}}$ associated with $B_{p,q}$. Note that $B_{p,q}$ linearly depends on the bonds B_p, ..., B_q. From this it easily follows $B_{p,q} > 0$ and $HB_{p,q}$ is a martingale. With that observations we conclude that $B_{p,q}$ is a numéraire as defined in Theorem 15.

Remark 25 *We focus on standard swaps here, meaning the fixed rate is settled on all dates of the underlying tenor structure of the Libors. For the treatment of the non-standard case see Schoenmakers [2005].*

The dynamics of swap rate process are given by (cf. Appendix A.8)

$$\begin{aligned} dS_{p,q} &= S_{p,q}\left(\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_p^T - \sigma_{j+1}^T\right) + \frac{B_q}{B_p - B_q}\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_p^T - \sigma_q^T\right)\right) dW^{p,q} \\ &=: S_{p,q}\sigma_{p,q}^T dW^{p,q} \end{aligned} \tag{1.28}$$

As mentioned above we will focus here on payer swaptions. Such an option gives the holder the right but not the obligation to enter at a fixed time point the underlying payer swap. The swap parameters like underlying, time, settlement date and so on are defined in the option contract. As stated earlier for the payer swap the holder pays the pre-fixed rate and receives the floating amount, if the swap is entered. It protects the holder against increasing interest rates. The contract maturing in T_p is equal to a cash-flow at T_p of

$$B_p\left(T_p\right)\left(Swap_{p,q,K}\left(T_p\right)\right)^+ .$$

Hence the value of such a contract for $t \leq T_p$ is given by

$$\begin{aligned} Swpn_{p,q,K}\left(t\right) &:= B_{p,q}\left(t\right) E_{p,q}^{\mathcal{F}_t}\left[\frac{\left(Swap_{p,q,K}\left(t\right)\right)^+}{B_{p,q}\left(T_p\right)}\right] \\ &= B_{p,q}\left(t\right) E_{p,q}^{\mathcal{F}_t}\left[\left(S_{p,q}\left(t\right) - K\right)^+\right]. \end{aligned} \tag{1.29}$$

As can be seen from (1.29) the swaption can be rewritten as a call option on the corresponding swap rate.

As in the Libor case we will point out a special situation where we are able to price swaptions in closed-form.

Definition 26 *If $t \mapsto \sigma_{p,q}\left(t\right)$ in (1.28) is a deterministic function we call the model for the swap rate (1.27) a swap market model (SMM).*

Proposition 27 *The price of a swaption in the SMM is given by*

$$B_{p,q}\left(t\right)\left(S_{p,q}\left(t\right)\mathcal{N}(d_+) - K\mathcal{N}(d_-)\right)$$

where $\mathcal{N}$ is the cumulative distribution function of the normal distribution and

$$d_\pm := \frac{\ln\left(S_{p,q}\left(t\right)/K\right) \pm \left(\sigma_{p,q}^B\right)^2\left(T_p - t\right)/2}{\sigma_{p,q}^B\sqrt{T_p - t}}, \quad \left(\sigma_{p,q}^B\right)^2 := \frac{1}{T_p - t}\int_t^{T_p}\left\|\sigma_{p,q}\left(s\right)\right\|^2 ds.$$

Proof. Since $\sigma_{p,q}$ is deterministic $S_{p,q}$ is a log-normal martingale under the measure $\mathbb{P}_{p,q}$. Further we have equality in distribution of

$$\int_t^{T_p}\sigma_{p,q}^T\left(s\right) dW^{p,q}\left(s\right) = \sigma_{p,q}^B\sqrt{T_p - t}\xi$$

for ξ being a standard normal distributed variable. The statement then follows from the Black-Scholes formula and

$$S_{p,q}\left(T_{p,q}\right) \overset{distr.}{=} S_{p,q}\left(t\right) \exp\left(-\frac{1}{2}\int\limits_{t}^{T_p} \left\|\sigma_{p,q}\left(s\right)\right\|^2 ds + \sigma_{p,q}^{B}\sqrt{T_p - t}\xi\right).$$

■

2. A new multi-factor stochastic volatility model with displacement

In this Chapter we establish an extended Libor model which is able to fit to caplet and swaption prices over a broad panel of strikes and maturities. We develop an approximated quasi-analytical pricing formula, involving the usage of FFT, for both, caplets and swaptions, which is crucial for an efficient calibration to market data. In a numerical study we show the ability of our model to produce good fits based on an only small approximation error.

2.1. Libor modeling in a new setting

The general representation (0.6) for the Libor dynamics will now be structured towards a multi-factor stochastic volatility model of type (0.5). Let us take

$$\Gamma_j = \begin{bmatrix} \sqrt{v_j}\widetilde{\beta}_j \\ \widetilde{\gamma}_j \\ 0 \end{bmatrix}, \quad \mathcal{W}^{(n)} = \begin{bmatrix} W^{(n)} \\ \widehat{W}^{(n)} \\ \overline{W}^{(n)} \end{bmatrix}, \quad \text{where}$$

$$dv_j = \kappa_j(\theta_j - v_j)dt + \sqrt{v_j}\left(\sigma_j^\top dW^{(n)} + \overline{\sigma}_j^\top d\overline{W}^{(n)}\right), \quad v_j(0) = \theta_j, \tag{2.1}$$

where $W^{(n)}$, $\widehat{W}^{(n)}$, $\overline{W}^{(n)}$ are mutually independent standard Brownian motions with dimensions m, $\widehat{m}$, and, $\overline{m}$, respectively, with $m + \widehat{m} + \overline{m} = d$. Further, for $1 \leq j < n$, $\widetilde{\beta}_j$ and $\widetilde{\gamma}_j$ are loading factors (in $\mathbb{R}^m$ and $\mathbb{R}^{\widehat{m}}$ respectively) to be specified below, and v_j are square-root volatility processes with parameters κ_j (mean reversion speed), θ_j (mean reversion level), and σ and $\overline{\sigma}$ are deterministic "vol of vol" factor loadings (in $\mathbb{R}^{\widehat{m}}$ and $\mathbb{R}^{\overline{m}}$ respectively), where (for convenience)

$$|\sigma_j|^2 + |\overline{\sigma}_j|^2 =: \varepsilon_j^2. \tag{2.2}$$

We thus get

$$\begin{aligned} \frac{dL_j}{L_j} = &- \sum_{k=j+1}^{n-1} \frac{\delta_k L_k}{1 + \delta_k L_k}\left(\widetilde{\beta}_j^\top \widetilde{\beta}_k \sqrt{v_j v_k} + \widetilde{\gamma}_j^\top \widetilde{\gamma}_k\right) dt \\ &+ \sqrt{v_j}\widetilde{\beta}_j^\top dW^{(n)} + \widetilde{\gamma}_j^\top d\widehat{W}^{(n)}, \end{aligned} \tag{2.3}$$

together with (2.1). We next set

$$\widetilde{\gamma}_j = \frac{L_j + \alpha_j}{L_j}\gamma_j, \quad \widetilde{\beta}_j = \frac{L_j + \alpha_j}{L_j}\beta_j, \tag{2.4}$$

for deterministic loading factors β_j and γ_j (in $\mathbb{R}^m$ and $\mathbb{R}^{\widehat{m}}$ respectively), and displacement constants α_j, $1 \leq j < n$, and we obtain from (2.3),

$$\begin{aligned}\frac{dL_j}{L_j + \alpha_j} = &- \sum_{k=j+1}^{n-1} \frac{\delta_k(L_k + \alpha_k)}{1 + \delta_k L_k} \left(\beta_j^\top \beta_k \sqrt{v_j v_k} + \gamma_j^\top \gamma_k\right) dt \\ &+ \sqrt{v_j}\beta_j^\top dW^{(n)} + \gamma_j^\top d\widehat{W}^{(n)},\end{aligned} \tag{2.5}$$

i.e. the new multi-factor stochastic volatility Libor model with displacement and stochastic volatilities driven by (2.1). By applying Ito's formula to the log-Libors, (2.5) becomes

$$\begin{aligned}d\ln\left(L_j + \alpha_j\right) = &-\frac{1}{2}\left|\gamma_j\right|^2 dt - \frac{1}{2}v_j\left|\beta_j\right|^2 dt \\ &- \sum_{k=j+1}^{n-1} \frac{\delta_k(L_k + \alpha_k)}{1 + \delta_k L_k} \left(\gamma_j^\top \gamma_k + \beta_j^\top \beta_k \sqrt{v_j v_k}\right) dt \\ &+ \sqrt{v_j}\beta_j^\top dW^{(n)} + \gamma_j^\top d\widehat{W}^{(n)}.\end{aligned} \tag{2.6}$$

In Section 2.2 we propose a pragmatic approximation that allows for quasi-analytical caplet pricing in the context of to (2.6).

2.1.1. Instantaneous correlations

For the mutual instantaneous Libor correlations we have

$$\begin{aligned}\mathrm{Cor}_{L_j, L_{j'}} := \frac{\frac{dL_j}{L_j} \cdot \frac{dL_{j'}}{L_{j'}}}{\sqrt{\frac{dL_j}{L_j} \cdot \frac{dL_j}{L_j}}\sqrt{\frac{dL_{j'}}{L_{j'}} \cdot \frac{dL_{j'}}{L_{j'}}}} &= \frac{\widetilde{\gamma}_j^\top \widetilde{\gamma}_{j'} + \sqrt{v_j v_{j'}}\widetilde{\beta}_j^\top \widetilde{\beta}_{j'}}{\sqrt{|\widetilde{\gamma}_j|^2 + v_j|\widetilde{\beta}_j|^2}\sqrt{|\widetilde{\gamma}_{j'}|^2 + v_{j'}|\widetilde{\beta}_{j'}|^2}} \\ &= \frac{\gamma_j^\top \gamma_{j'} + \sqrt{v_j v_{j'}}\beta_j^\top \beta_{j'}}{\sqrt{|\gamma_j|^2 + v_j|\beta_j|^2}\sqrt{|\gamma_{j'}|^2 + v_{j'}|\beta_{j'}|^2}},\end{aligned}$$

which yields for $\gamma \equiv 0$, $\mathrm{Cor}_{L_j,L_{j'}} = \frac{\beta_j^\top \beta_{j'}}{|\beta_j||\beta_{j'}|}$, and for $\beta \equiv 0$, $\mathrm{Cor}_{L_j,L_{j'}} = \frac{\gamma_j^\top \gamma_{j'}}{|\gamma_j||\gamma_{j'}|}$ as usual. For the instantaneous correlations between Libors and the stochastic volatilities we have

$$\begin{aligned}
\mathrm{Cor}_{L_j,v_{j'}} &:= \frac{\frac{dL_j}{L_j} \cdot dv_{j'}}{\sqrt{\frac{dL_j}{L_j} \cdot \frac{dL_j}{L_j}}\sqrt{dv_{j'} \cdot dv_{j'}}} = \frac{\sqrt{v_j v_{j'}}\widetilde{\beta}_j^\top \sigma_{j'}}{\sqrt{|\widetilde{\gamma}_j|^2 + v_j|\widetilde{\beta}_j|^2}\sqrt{v_{j'}(|\sigma_{j'}|^2 + |\overline{\sigma}_{j'}|^2)}} \qquad (2.7) \\
&= \frac{\sqrt{v_j}\beta_j^\top \sigma_{j'}}{\sqrt{|\gamma_j|^2 + v_j|\beta_j|^2}\varepsilon_{j'}}.
\end{aligned}$$

For $\gamma \equiv 0$ we thus obtain

$$\mathrm{Cor}_{L_j,v_{j'}} = \frac{\beta_j^\top \sigma_{j'}}{|\beta_j|\varepsilon_{j'}}.$$

For the mutual instantaneous correlations between the stochastic volatilities we get

$$\mathrm{Cor}_{v_j,v_{j'}} := \frac{dv_j \cdot dv_{j'}}{\sqrt{dv_j \cdot dv_j}\sqrt{dv_{j'} \cdot dv_{j'}}} = \frac{\sigma_j^\top \sigma_{j'} + \overline{\sigma}_j^\top \overline{\sigma}_{j'}}{\varepsilon_j \varepsilon_{j'}}.$$

2.1.2. Discussion of the Wu-Zhang model as a special case

Let us take as a special case $\gamma \equiv 0$, $\alpha_j \equiv 0$, $\kappa_j \equiv \kappa_0$, $\theta_j \equiv \theta_0$, and $\sigma_j \equiv (0, ..., 0, \sigma_0) \in \mathbb{R}^m$. That is, we basically have only one single volatility process driven by a one-dimensional Brownian motion (i.e. $\overline{\sigma} = 0$). Further we set for each j,

$$\beta_j := |\beta_j| \begin{pmatrix} \sqrt{1-\rho_j^2} e_j \\ \rho_j \end{pmatrix} \in \mathbb{R}^m,$$

where $e_j \in \mathbb{R}^{m-1}$ is a unit vector, $|\beta_j|$ is deterministic, and $-1 \leq \rho_j \leq 1$ denotes a (possibly time dependent) deterministic correlation (function). We now are in the setting of Wu and Zhang [2006]. Let us consider in this setting the mutual Libor correlations, being

$$\begin{aligned}
\mathrm{Cor}_{L_j,L_{j'}} &:= \frac{\frac{dL_j}{L_j} \cdot \frac{dL_{j'}}{L_{j'}}}{\sqrt{\frac{dL_j}{L_j} \cdot \frac{dL_j}{L_j}}\sqrt{\frac{dL_{j'}}{L_{j'}} \cdot \frac{dL_{j'}}{L_{j'}}}} = \frac{\beta_j^\top \beta_{j'}}{|\beta_j||\beta_{j'}|} \qquad (2.8) \\
&= \sqrt{1-\rho_j^2}\sqrt{1-\rho_{j'}^2} e_j^\top e_{j'} + \rho_j \rho_{j'}.
\end{aligned}$$

Clearly, (2.8) imposes a severe restriction on the mutual Libor correlation structure and the stochastic volatility correlations. For example, if ρ_j is relatively large, let us say about 0.7, and $e_j^\top e_{j'}$ is taken according to some suitably parameterized correlation structure (e.g. see Schoenmakers [2005]), then (2.8) is bounded from below by 0.5. In contrast, as a main feature of the multi-factor model (2.1)-(2.6), we have full flexibility regarding the stochastic volatility correlations ρ_j, and the mutual Libor correlations (2.8).

Remark 28 *If $\alpha_j \equiv 0$, a Libor market model is retrieved by taking $\beta_j \equiv 0$, or by taking $v_j(0) = \theta_j \equiv 1$, $\sigma_j \equiv \overline{\sigma}_j \equiv 0$. A further reason for including the LMM term $\gamma_j^\top d\widehat{W}$ in the Libor noise might be to have some extra freedom for calibrating to swaptions due to the fact that caplet prices only depend on $|\gamma_j|$.*

2.2. Approximate caplet pricing and calibration

For quasi-analytical caplet pricing we will construct an (approximate) characteristic function of L_j under P_{j+1}. Let us write (2.5) as

$$\begin{aligned}\frac{dL_j}{L_j + \alpha_j} &= \sqrt{v_j}\beta_j^\top \left[dW^{(n)} - \sum_{k=j+1}^{n-1} \frac{\delta_k(L_k + \alpha_k)}{1 + \delta_k L_k} \beta_k \sqrt{v_k} dt \right] \\ &+ \gamma_j^\top \left[d\widehat{W}^{(n)} - \sum_{k=j+1}^{n-1} \frac{\delta_k(L_k + \alpha_k)}{1 + \delta_k L_k} \gamma_k dt \right] \\ &=: \sqrt{v_j}\beta_j^\top dW^{(j+1)} + \gamma_j^\top d\widehat{W}^{(j+1)}.\end{aligned}$$

Since L_j is a martingale under P_{j+1}, we necessarily have that $dW^{(j+1)}$ and $d\widehat{W}^{(j+1)}$ are standard Brownian motions under P_{j+1}. Since the covariation processes $\langle \overline{W}^{(n)}, B_j \rangle \equiv 0$ for all j, it follows that $d\overline{W}^{(j+1)} = d\overline{W}^{(n)}$ for all j (cf. Wu and Zhang [2006] and Belomestny, Mathew and Schoenmakers [2011]). The dynamics of the stochastic volatility process v_j under P_{j+1} can thus be written as

$$\begin{aligned} dv_j &= \kappa_j(\theta_j - v_j)dt + \sqrt{v_j}\overline{\sigma}_j^\top d\overline{W}^{(j+1)} \\ &+ \sqrt{v_j}\sigma_j^\top \left[dW^{(j+1)} + \sum_{k=j+1}^{n-1} \frac{\delta_k(L_k + \alpha_k)}{1 + \delta_k L_k} \beta_k \sqrt{v_k} dt \right] \\ &= \underbrace{\left(\kappa_j(\theta_j - v_j) + \sum_{k=j+1}^{n-1} \frac{\delta_k(L_k + \alpha_k)}{1 + \delta_k L_k} \sigma_j^\top \beta_k \sqrt{v_j v_k} \right) dt}_{(*)} \\ &+ \sqrt{v_j} \left(\sigma_j^\top dW^{(j+1)} + \overline{\sigma}_j^\top d\overline{W}^{(j+1)} \right). \end{aligned}$$

Thus, in order to obtain approximate affine dynamics for v_j it is enough to approximate $(*)$ with an expression that is affine in v_j. Let us therefore consider the pragmatic approximation

$$\sqrt{v_j v_k} = \sqrt{v_j \frac{v_k E v_j}{E v_j}} \approx \sqrt{v_j \frac{v_j E v_k}{E v_j}} \approx v_j \sqrt{\frac{\theta_k}{\theta_j}} \tag{2.9}$$

(note that $E\, v_k = \theta_k$ due to the initial condition in (2.1)). In the Wu-Zhang setting we have $v_j \equiv v$ and thus, strict equality in (2.9) appears. Combining (2.9) and usual freezing of Libors in $(*)$ then leads to the following approximate volatility dynamics,

$$dv_j \approx \kappa_j \theta_j dt + \left(-\kappa_j + \sum_{k=j+1}^{n-1} \sqrt{\frac{\theta_k}{\theta_j}} \left[\frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} \right] (0) \sigma_j^\top \beta_k \right) v_j dt$$
$$+ \sqrt{v_j} \left(\sigma_j^\top dW^{(j+1)} + \overline{\sigma}_j^\top d\overline{W}^{(j+1)} \right).$$

With

$$\kappa_j^{(j+1)} = \kappa_j - \sum_{k=j+1}^{n-1} \sqrt{\frac{\theta_k}{\theta_j}} \left[\frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} \right] (0) \sigma_j^\top \beta_k \tag{2.10}$$
$$\theta_j^{(j+1)} = \frac{\kappa_j \theta_j}{\kappa_j^{(j+1)}}$$

we thus obtain from (2.6) the approximative system

$$d \ln (L_j + \alpha_j) = -\frac{1}{2} |\gamma_j|^2 dt - \frac{1}{2} v_j |\beta_j|^2 dt + \sqrt{v_j} \beta_j^\top dW^{(j+1)} + \gamma_j^\top d\widehat{W}^{(j+1)}, \tag{2.11}$$
$$dv_j = \kappa_j^{(j+1)} \left(\theta_j^{(j+1)} - v_j \right) dt + \sqrt{v_j} \left(\sigma_j^\top dW^{(j+1)} + \overline{\sigma}_j^\top d\overline{W}^{(j+1)} \right), \quad v_j(0) = \theta_j.$$

Now the main point is that, if moreover β_j, σ_j, and $\overline{\sigma}_j$ are constant in time (piece-wise constant would be enough in fact), (2.11) is an affine structure that allows for Fourier based (approximate) caplet pricing.

2.2.1. Caplet pricing via characteristic function

In general the price of a T_j-caplet with strike K is given by

$$\begin{aligned} C_j(K) &= \delta_j B_{j+1}(0) E_{j+1} (L_j(T_j) - K)^+ \\ &= B_{j+1}(0) \delta_j E_{j+1} (L_j(T_j) + \alpha_j - (K + \alpha_j))^+ \\ &= B_{j+1}(0) \delta_j E_{j+1} ((L_j(0) + \alpha_j) \, e^{\ln \frac{L_j(T_j) + \alpha_j}{L_j(0) + \alpha_j}} - (K + \alpha_j))^+ \\ &=: B_{j+1}(0) \delta_j E_{j+1} (L_j^{disp}(0) e^{\ln \frac{L_j^{disp}(T_j)}{L_j^{disp}(0)}} - K_j^{disp})^+. \end{aligned}$$

We may thus apply the Carr-Madan Fourier pricing method (outlined in the next Subsection) for caplets using

$$\varphi_{j+1}^{disp}, \quad \widehat{L}_j^{disp}(0), \quad K_j^{disp},$$

where the characteristic function

$$\varphi_{j+1}^{disp}(z\,;v) := E_{j+1}\left[e^{\mathrm{i}z\ln\frac{L_j^{disp}(T_j)}{L_j^{disp}(0)}} \middle| v_j(0) = v \right] \tag{2.12}$$

may be obtained as follows. Let us abbreviate for fixed j, $X^{0,x,v}(t) := \ln L_j^{disp}(t) = \ln(L_j(t) + \alpha_j)$ with $X^{0,x,v}(0) = \ln L_j^{disp}(0) = \ln(L_j(0) + \alpha_j) =: x$, and $V^{0,x,v}(t) := v_j(t)$ with $V^{0,x,v}(0) = v_j(0) =: v$. Then by (2.11) (using (2.2)), the generator of the vector process (X, V) is given by

$$\begin{aligned} A := A_{x,v} := & \left(-\frac{1}{2}\left|\gamma_j\right|^2 - \frac{1}{2}v\left|\beta_j\right|^2\right)\frac{\partial}{\partial x} + \kappa_j^{(j+1)}\left(\theta_j^{(j+1)} - v\right)\frac{\partial}{\partial v} \\ & + \frac{1}{2}\left(\left|\gamma_j\right|^2 + v\left|\beta_j\right|^2\right)\frac{\partial^2}{\partial x^2} + v\sigma_j^\top\beta_j\frac{\partial^2}{\partial x\partial v} + \frac{1}{2}\varepsilon_j^2 v\frac{\partial^2}{\partial v^2}. \end{aligned}$$

Let $\widehat{p}\,(z, z'\,; t, x, v)$ satisfy the Cauchy initial value problem

$$\frac{\partial\widehat{p}}{\partial t} = A\widehat{p}, \qquad \widehat{p}(z, z'\,; 0, x, v) = e^{\mathrm{i}(zx+z'v)}. \tag{2.13}$$

Then

$$\widehat{p}\,(z, z'\,; t, x, v) = Ee^{\mathrm{i}\left(zX^{0,x,v}(t)+z'V^{0,x,v}(t)\right)}.$$

We are only interested in the solution for $z' = 0$. Let us therefore consider the ansatz

$$\widehat{p}\,(z\,; t, x, v) = \exp\left(A(z;t) + B_0(z;t)x + B(z;t)v\right)$$

with

$$A(z;0) = 0, \quad B_0(z;0) = \mathrm{i}z, \quad B(z;0) = 0. \tag{2.14}$$

Substitution in (2.13) yields,

$$\begin{aligned} \left(\frac{\partial A}{\partial t} + \frac{\partial B_0}{\partial t}x + \frac{\partial B}{\partial t}v\right) = & \left(-\frac{1}{2}\left|\gamma_j\right|^2 - \frac{1}{2}v\left|\beta_j\right|^2\right)B_0 \\ & + \kappa_j^{(j+1)}\left(\theta_j^{(j+1)} - v\right)B + \frac{1}{2}\left(\left|\gamma_j\right|^2 + v\left|\beta_j\right|^2\right)B_0^2 \\ & + v\sigma_j^\top\beta_j B_0 B + \frac{1}{2}\varepsilon_j^2 v B^2, \end{aligned}$$

and we get the Riccati system

$$\begin{aligned}\frac{\partial A}{\partial t} &= -\frac{1}{2}\left|\gamma_j\right|^2 B_0 + \kappa_j^{(j+1)}\theta_j^{(j+1)} B + \frac{1}{2}\left|\gamma_j\right|^2 B_0^2 \\ \frac{\partial B_0}{\partial t} &= 0 \\ \frac{\partial B}{\partial t} &= -\frac{1}{2}\left|\beta_j\right|^2 B_0 - \kappa_j^{(j+1)} B + \frac{1}{2}\left|\beta_j\right|^2 B_0^2 + \sigma_j^\top \beta_j B_0 B + \frac{1}{2}\varepsilon_j^2 B^2.\end{aligned}$$

Taking into account (2.14) we get

$$\begin{aligned}\frac{\partial A}{\partial t} &= -\frac{1}{2}\left|\gamma_j\right|^2 \left(\mathrm{i}z + z^2\right) + \kappa_j^{(j+1)}\theta_j^{(j+1)} B \\ \frac{\partial B}{\partial t} &= -\frac{1}{2}\left|\beta_j\right|^2 \left(\mathrm{i}z + z^2\right) - \left(\kappa_j^{(j+1)} - \mathrm{i}z\sigma_j^\top \beta_j\right) B + \frac{1}{2}\varepsilon_j^2 B^2.\end{aligned}$$

It is well known (see Heston [1993]) that this system can be explicitly solved, but depending on the chosen branch of the complex logarithm one may have different representations for its solution. We follow Lord and Kahl's representation due to the principal branch, see Lord and Kahl [2010][1], and obtain

$$B(z;t) = \frac{a_j + d_j}{\varepsilon_j^2}\frac{1 - e^{d_j t}}{1 - g_j e^{d_j t}}$$

and

$$A(z;t) = -\frac{1}{2}\left(\mathrm{i}z + z^2\right)\int_0^t \left|\gamma_j\right|^2 ds + \frac{\kappa_j^{(j+1)}\theta_j^{(j+1)}}{\varepsilon_j^2}\left\{\left(a_j - d_j\right)t - 2\ln\left[\frac{e^{-d_j t} - g_j}{1 - g_j}\right]\right\}$$

with

$$\begin{aligned}a_j &= \kappa_j^{(j+1)} - \mathrm{i}z\sigma_j^\top \beta_j \\ d_j &= \sqrt{a_j^2 + \left|\beta_j\right|^2 \left(\mathrm{i}z + z^2\right)\varepsilon_j^2} \\ g_j &= \frac{a_j + d_j}{a_j - d_j}.\end{aligned}$$

Resuming, by taking $t = T_j$ we get for (2.12),

$$\begin{aligned}\varphi_{j+1}^{disp}(z\,;v) &= e^{-\mathrm{i}z\ln L_j^{disp}(0)}\widehat{p}\left(z\,;T_j, \ln L_j^{disp}(0), v\right) \\ &= \exp\left(\widetilde{A}(z;T_j) + B(z;T_j)v\right)\exp\left(-\frac{1}{2}\left(\mathrm{i}z + z^2\right)\int_0^{T_j}\left|\gamma_j\right|^2 ds\right)\end{aligned} \tag{2.15}$$

[1] In a personal communication, Roger Lord confirmed a typo in the published version and so referred to the preprint version.

with

$$B(z;T_j) = \frac{a_j + d_j}{\varepsilon_j^2} \frac{1 - e^{d_j T_j}}{1 - g_j e^{d_j T_j}}, \quad \text{and}$$

$$\widetilde{A}(z;t) := \frac{\kappa_j^{(j+1)} \theta_j^{(j+1)}}{\varepsilon_j^2} \left\{ (a_j - d_j) T_j - 2 \ln \left[\frac{e^{-d_j T_j} - g_j}{1 - g_j} \right] \right\}.$$

2.2.2. Carr & Madan inversion formula

Following Carr and Madan [1999], the T_j-caplet price is now obtained by the inversion formula,

$$C_j(K) = \delta_j B_{j+1}(0)(L_j^{disp}(0) - K_j^{disp})^+ + \frac{\delta_j B_{j+1}(0) L_j^{disp}(0)}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \varphi_{j+1}^{disp}(z - \mathfrak{i}; \theta_j)}{z(z - \mathfrak{i})} e^{-\mathrm{i}z \ln \frac{K_j^{disp}}{L_j^{disp}(0)}} dz, \tag{2.16}$$

where φ_{j+1}^{disp} is given by (2.15) and we recall that $v_j(0) = \theta_j$. The integrand in (2.16) decays with order z^{-2} if $|z| \to \infty$, which is rather slow from a numerical point of view. It is therefore advantageous to modify the inversion formula in the following way. Let $\varphi_{j+1}^{\mathcal{B},disp}$ be the characteristic function (2.12) due to the Black model,

$$L_j^{disp}(T_j) = L_j^{disp}(0) e^{-\frac{1}{2}(\sigma^B)^2 T_j + \sigma^B \sqrt{T_j} \varsigma}, \quad \varsigma \in N(0,1)$$

in the measure P_{j+1}, with a certain suitably chosen volatility σ_j^B. We then have (cf. Black's 76 formula)

$$E_{j+1} \left(L_j^{disp}(T_j) - K^{disp} \right)^+ = \mathcal{B}(L_j^{disp}(0), T_j, \sigma^B, K^{disp}),$$

where

$$\mathcal{B}(L, T, \sigma, K) := L\mathcal{N}(d_+) - K\mathcal{N}(d_-), \quad \text{with}$$

$$d_\pm := \frac{\ln \frac{L}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}, \quad \text{and}$$

$$\varphi_{j+1}^{\mathcal{B},disp}(z\,; v) = \varphi_{j+1}^{\mathcal{B},disp}(z) = E_{j+1} e^{\mathrm{i}z\left(-\frac{1}{2}(\sigma^B)^2 T_j + \sigma^B \sqrt{T_j}\varsigma\right)} = e^{-\frac{1}{2}(\sigma^B)^2 T_j (z^2 + \mathrm{i}z)}.$$

Now applying Carr and Madan's formula to the Black model yields

$$C_j^{\mathcal{B}}(K) := \delta_j B_{j+1}(0)\mathcal{B}(L_j^{disp}(0), T_j, \sigma^B, K_j^{disp}) = \delta_j B_{j+1}(0)(L_j^{disp}(0) - K_j^{disp})^+ \quad (2.17)$$
$$+ \frac{\delta_j B_{j+1}(0) L_j^{disp}(0)}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \varphi_{j+1}^{\mathcal{B},disp}(z - \mathfrak{i})}{z(z - \mathfrak{i})} e^{-\mathrm{i}z \ln \frac{K_j^{disp}}{L_j^{disp}(0)}} dz,$$

and by subtracting (2.17) from (2.16) we get

$$C_j(K) = C_j^{\mathcal{B}}(K) + \quad (2.18)$$
$$\frac{\delta_j B_{j+1}(0) L_j^{disp}(0)}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi_{j+1}^{\mathcal{B},disp}(z - \mathfrak{i}; v) - \varphi_{j+1}^{disp}(z - \mathfrak{i}; \theta_j)}{z(z - \mathfrak{i})} e^{-\mathrm{i}z \ln \frac{K_j^{disp}}{L_j^{disp}(0)}} dz.$$

The latter inversion formula is usually much more efficient since typically the integrand decays much faster than in (2.16).

2.2.3. Putting the caplet approximation to the test

We now test the accuracy of the Fourier based caplet pricing method (2.18) via the approximative characteristic function (2.15). In this respect we compare, for each particular j, the simulation price of the "true" model (2.5) with the simulation price due to the model obtained by replacing each volatility dynamics v_k, $k \neq j$, with the process v_j, yielding a Wu-Zhang type approximation depending on j in fact. In turn, the Fourier based T_j-caplet price approximation is known to be a very accurate approximation to the j-linked Wu-Zhang model, as already documented in Wu and Zhang [2006].

The initial Libor rates are stripped from a given spot interest rate curve (see Table 2.1). In the test model we drop the Gaussian part, i.e. $\gamma_j \equiv 0$, and also assume that no displacement is in force, i.e. $\alpha \equiv 0$. We choose $\delta_j = T_{j+1} - T_j \equiv 1.0$ and we put (2.1) and (2.5) according to Subsection 2.2.4, where

$$\beta_j = 0.15 e_j, \quad \text{such that} \qquad r_{ij} = e_i^{\top} e_j = e^{-0.073|T_i - T_j|}, \quad (2.19)$$

and the other parameters are given in Table 2.1. The orthonormal vectors e_j are obtained by a Cholesky decomposition of the correlation matrix (r_{ij}). The parameters for the stochastic volatility processes are taken to be representative for a typical calibration. In particular they are chosen in such a way that the Feller condition $2\kappa\theta > \sigma^2$ is violated. The mean reversion levels are uniformly set to $\theta_j \equiv 1$. We compare caplet prices due to the "true" model and the approximative one, by Monte Carlo simulation based on $30,000$ simulated paths (Table 2.2).

The numerical results show that (2.11) approximates very accurately the true model dynamics (2.1) and (2.5). Indeed, the absolute price deviations are of magnitudes within basis points, with a well behaved relative error for ITM (in-the-money) and ATM (at-the-money) contracts. The relative errors become somewhat larger for OTM contracts, but OTM (out-of-the-money) caplet prices are typically very low (close to worthlessness) so

j	ρ_j	κ_j	ε_j	$B_j(0)$	$L_j(0)$
1	-0.70	4.00000000	3.00000000	0.971717	0.0332468
2	-0.70	3.95918367	2.97959184	0.94045	0.0257067
3	-0.70	3.91836735	2.95918367	0.91688	0.0195338
4	-0.70	3.87755102	2.93877551	0.899313	0.0235296
5	-0.70	3.83673469	2.91836735	0.878639	0.0278511
6	-0.70	3.79591837	2.89795918	0.854831	0.0258653
7	-0.70	3.75510204	2.87755102	0.833278	0.02359
8	-0.70	3.71428571	2.85714286	0.814074	0.0237439
9	-0.70	3.67346939	2.83673469	0.795193	0.0240497
10	-0.70	3.63265306	2.81632653	0.776518	0.023694
11	-0.70	3.59183673	2.79591837	0.758545	0.0234799
12	-0.70	3.55102041	2.77551020	0.741143	0.0236513
13	-0.70	3.51020408	2.75510204	0.724019	0.0238636
14	-0.70	3.46938776	2.73469388	0.707144	0.0240064
15	-0.70	3.42857143	2.71428571	0.690566	0.0241881
16	-0.70	3.38775510	2.69387755	0.674257	0.0244311
17	-0.70	3.34693878	2.67346939	0.658177	0.0246647
18	-0.70	3.30612245	2.65306122	0.642334	0.024855
19	-0.70	3.26530612	2.63265306	0.626756	0.0249485

Table 2.1.: Parameters of the Libor model, present values and initial Libor rates, terminal bond $B_{20}(0) = 0.6115$.

that relative errors stemming from approximation (2.9), (2.11) are intrinsically unstable (for any "good" approximation in fact).

2.2.4. Further structuring and calibration

As part of the model, we choose a fixed LMM part γ_j of the Libor structure. This part may be obtained from an LMM calibration, eventually weighted with some factor for instance or, if enough flexibility is left for our purposes, we may set $\gamma_j \equiv 0$. The loadings β_j are also assumed to be chosen in advance. We further take $\overline{m} = 1$ in (2.1), and for ρ_j, $-1 \leq \rho_j \leq 1$, we take $\sigma_j =: \varepsilon_j \rho_j e_j$, where $\beta_j =: |\beta_j| e_j$, and so $\overline{\sigma}_j =: \sqrt{1-\rho_j^2}\varepsilon_j$. Note that in principle we have no restrictions on ρ_j conferred to the Wu-Zhang case (see Subsection 2.1.2). Then (2.7) becomes

$$\mathrm{Cor}_{L_j, v_{j'}} = \rho_{j'} e_j^\top e_{j'} = \rho_{j'} r_{jj'}$$

with $r_{jj'} := e_j^\top e_{j'}$, and in particular we have $\mathrm{Cor}_{L_j, v_j} = \rho_j$. For the mutual correlations between the volatility processes we so have

$$\mathrm{Cor}_{v_j, v_{j'}} = \rho_j \rho_{j'} r_{jj'} + \sqrt{1-\rho_j^2}\sqrt{1-\rho_{j'}^2}. \tag{2.20}$$

T_j	Strike	Price (SE)	Approx. price (SE)	Abs. error	Rel. error
5.0	0.000	0.0245 (9.28e-05)	0.0244 (9.00e-05)	1.71e-04	0.0069
	0.005	0.0201 (8.96e-05)	0.0200 (8.68e-05)	1.66e-04	0.0082
	0.010	0.0158 (8.62e-05)	0.0156 (8.34e-05)	1.64e-04	0.0104
	0.015	0.0115 (8.12e-05)	0.0113 (7.85e-05)	1.75e-04	0.0151
	0.020	0.0076 (7.25e-05)	0.0075 (7.00e-05)	1.97e-04	0.0255
	0.025	0.0045 (5.96e-05)	0.0043 (5.72e-05)	2.03e-04	0.0445
	0.030	0.0023 (4.45e-05)	0.0022 (4.20e-05)	1.74e-04	0.0729
11.0	0.000	0.0179 (9.91e-05)	0.0177 (9.45e-05)	2.50e-04	0.0139
	0.005	0.0141 (9.61e-05)	0.0139 (9.15e-05)	2.45e-04	0.0173
	0.010	0.0105 (9.16e-05)	0.0102 (8.72e-05)	2.56e-04	0.0243
	0.015	0.0073 (8.36e-05)	0.0070 (7.94e-05)	2.74e-04	0.0375
	0.020	0.0047 (7.24e-05)	0.0045 (6.82e-05)	2.73e-04	0.0571
	0.025	0.0029 (5.97e-05)	0.0027 (5.56e-05)	2.45e-04	0.0823
	0.030	0.0018 (4.85e-05)	0.0016 (4.44e-05)	1.99e-04	0.109
15.0	0.000	0.0168 (1.06e-04)	0.0165 (1.00e-04)	2.81e-04	0.0166
	0.005	0.0134 (1.04e-04)	0.0131 (9.86e-05)	2.79e-04	0.0208
	0.010	0.0101 (1.00e-04)	0.0098 (9.48e-05)	2.95e-04	0.0290
	0.015	0.0074 (9.29e-05)	0.0070 (8.76e-05)	3.14e-04	0.0423
	0.020	0.0052 (8.31e-05)	0.0049 (7.78e-05)	3.14e-04	0.0602
	0.025	0.0035 (7.22e-05)	0.0033 (6.69e-05)	2.92e-04	0.0813
	0.030	0.0024 (6.14e-05)	0.0021 (5.62e-05)	2.53e-04	0.1043
19.0	0.000	0.0158 (1.03e-04)	0.0155 (9.81e-05)	2.74e-04	0.0172
	0.005	0.0127 (1.03e-04)	0.0124 (9.74e-05)	2.77e-04	0.0217
	0.010	0.0098 (1.00e-04)	0.0095 (9.46e-05)	2.98e-04	0.0302
	0.015	0.0074 (9.43e-05)	0.0071 (8.88e-05)	3.19e-04	0.0430
	0.020	0.0055 (8.62e-05)	0.0051 (8.08e-05)	3.25e-04	0.0509
	0.025	0.0040 (7.72e-05)	0.0037 (7.17e-05)	3.12e-04	0.0773
	0.030	0.0029 (6.81e-05)	0.0026 (6.26e-05)	2.84e-04	0.0963

Table 2.2.: Simulation results for caplets.

In any case the scalars $\kappa_j, \theta_j, \rho_j, \varepsilon_j$, and the loadings have to be time independent, in order to invoke standard square-root volatility processes. In principle piece-wise constant $t \mapsto \beta_j(t)$ will allow for Fourier based caplet pricing later on, but for simplicity we assume henceforth that the β_j are also time independent.

Remark 29 *In practice it turns out that the ρ_j are negative overall in order to produce a skew. Let us assume for simplicity that we could fit the data with a uniform (negative or positive) ρ. Then (2.20) implies $Cor_{v_j,v_{j'}} = 1 - \rho^2(1 - r_{jj'}) \geq 1 - \rho^2$, assuming that mutual Libor correlations $r_{jj'}$ are non-negative. This means that mutual correlations between volatility processes are typically high (≥ 0.5 for $|\rho| = 0.7$), and even close to 1 when j' is close to j.*

2.2.5. Calibration to caplet volatility-strike-maturity

We will now illustrate a typical calibration test of the stochastic volatility Libor model in its terminal measure to market cap-strike data. The test is carried out for EurIBOR market data from September 20, 2010, based on a twenty year annual tenor structure. In Section 2.4 we will show calibration results on a semi-annual tenor structure applying the full calibration strength of our model. Here we stick to a simplified version to keep the simulation, carried out later on, tractable. For simplicity, the displacements and the Gaussian part where taken to be zero, i.e. $\alpha_j \equiv 0$, $\gamma_i \equiv 0$, and as further input parameters we took $\theta_i \equiv 1$, and e_i from a Cholesky decomposition according to $e_i^\top e_j = r_{ij} = e^{-0.118|T_i - T_j|}$. For each maturity T_j, the parameters

$$|\beta_j|, \ \kappa_j, \ \varepsilon_j, \ \rho_j,$$

where next calibrated to the caplet price-strike panel corresponding to T_j, obtained from the market data. This calibration involves a minimum search of a standard averaged relative error functional based on the FFT pricing formula (2.16) due to the characteristic function (2.15). Each trial κ_j (which is restricted to $\kappa_j > 0$) induces a $\kappa_j^{(j+1)}$ and $\theta_j^{(j+1)}$ via (2.10) (recall that $\theta_i \equiv 1$) which, together with ρ_j, are subsequently plugged into (2.15). The implied volatility patterns due to the calibration as well as the calibrated parameters are depicted in Figure 2.1. Concluding we may say that we obtained a satisfactory model fit with robustly behaving parameters when moving from one maturity to the other. Optically the fits for small strikes, hence deep ITM caplets may look a little bit off overall. However, this is only appearance because our algorithm calibrates to caplet *prices*, while implied volatilities are badly conditioned for deep ITM strikes.

2.3. Swap rate dynamics and approximate swaption pricing

2.3.1. Swap contracts and dynamics under swap measures

An interest rate swap is a contract to exchange a series of floating interest payments in return for a series of fixed rate payments. Consider a series of payment dates between T_{p+1} and T_q, $q > p$. At each time T_{j+1}, $j = p, \ldots, q-1$, the fixed leg of a (standard)

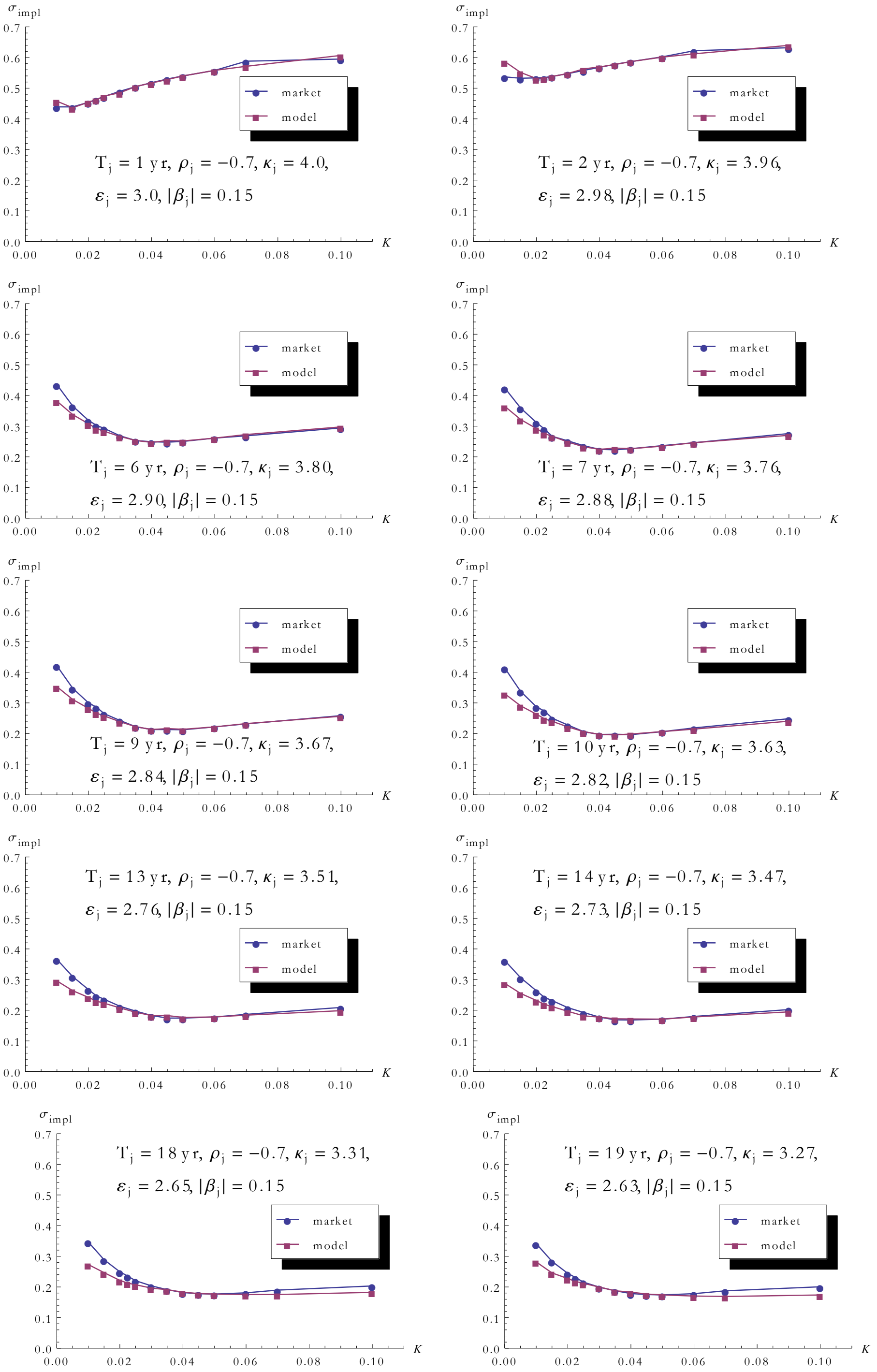

Figure 2.1.: Implied caplet volatilities due to market data vs. calibrated model

swap pays $\delta_j K$, whereas in return the floating leg pays $\delta_j L_j(T_j)$ with $L_j(T_j)$ being the spot Libor rate. Consequently, the time $t-$value of the interest rate swap (with $t \le T_p$) is

$$\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)(L_j(t) - K).$$

The swap rate $S_{p,q}(t)$ is defined to be the value of K for which the present value of the contract is zero. We thus have

$$S_{p,q}(t) = \frac{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t) L_j(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)} = \frac{B_p(t) - B_q(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)}. \tag{2.21}$$

So $S_{p,q}$ is a martingale under the probability measure $P_{p,q}$, induced by the annuity numéraire

$$B_{p,q}(t) := \sum_{j=p}^{q-1} \delta_j B_{j+1}(t).$$

From (2.5) it follows that

$$dS_{p,q}(t) = S_{p,q}(t) \Lambda_{p,q}^\top(t) d\mathcal{W}^{p,q}(t), \tag{2.22}$$

where $\mathcal{W}^{(p,q)} := (W^{p,q}, \widehat{W}^{p,q})$ is standard Brownian motion under $P_{p,q}$. However we will rewrite (2.22) and modify the approximation compared to the way it was done in Ladkau, Schoenmakers and Zhang [2013] to obtain a consistent swaption pricing formula in the sense that we get the same price for $S_{p,p+1} = L_p$. Therefore we look at

$$dS_{p,q}(t) = \frac{S_{p,q}(t) + \alpha_{p,q}}{S_{p,q}(t) + \alpha_{p,q}} S_{p,q}(t) \Lambda_{p,q}^\top(t) d\mathcal{W}^{p,q}(t),$$

where

$$\Lambda_{p,q} = \sum_{j=p}^{q-1} \frac{\delta_j \left(L_j + \alpha_j\right)}{1 + \delta_j L_j} \left(\sum_{l=j}^{q-1} w_l^{p,q} + \frac{B_q}{B_p - B_q} \right) \begin{bmatrix} \sqrt{v_j} \beta_j \\ \gamma_j \end{bmatrix}, \; w_l^{p,q} := \frac{\delta_l B_{l+1}}{B_{p,q}}. \tag{2.23}$$

The derivation hereof is given in Appendix A.8. We further have (see Appendix A.8),

$$d\mathcal{W}^{p,q} = d\mathcal{W}^{(n)} - dt \sum_{l=p}^{q-1} w_l^{p,q} \sum_{k=l+1}^{n-1} \frac{\delta_k \left(L_k + \alpha_k\right)}{1 + \delta_k L_k} \begin{bmatrix} \sqrt{v_k} \beta_k \\ \gamma_k \end{bmatrix}. \tag{2.24}$$

By (2.22) we thus get

$$\begin{aligned} d \ln S_{p,q} &= -\frac{1}{2} \frac{1}{S_{p,q}^2} d\langle S_{p,q} \rangle + \frac{dS_{p,q}}{S_{p,q}} \\ &= -\frac{1}{2} \left| \Lambda_{p,q} \right|^2 dt + \Lambda_{p,q}^\top d\mathcal{W}^{p,q}, \end{aligned} \tag{2.25}$$

$$d\ln\left(S_{p,q}+\alpha_{p,q}\right) = -\frac{1}{2}\frac{1}{\left(S_{p,q}+\alpha_{p,q}\right)^2}d\langle S_{p,q}\rangle + \frac{dS_{p,q}}{S_{p,q}+\alpha_{p,q}} \tag{2.26}$$
$$= -\frac{1}{2}\left|\frac{S_{p,q}}{S_{p,q}+\alpha_{p,q}}\Lambda_{p,q}\right|^2 dt + \frac{S_{p,q}}{S_{p,q}+\alpha_{p,q}}\Lambda_{p,q}^\top d\mathcal{W}^{p,q},$$

where by (2.23) we may write

$$\frac{S_{p,q}}{S_{p,q}+\alpha_{p,q}}\Lambda_{p,q} = \sum_{j=p}^{q-1}\begin{bmatrix}\sqrt{v_j}\beta_j \\ \gamma_j\end{bmatrix}\frac{L_j+\alpha_j}{S_{p,q}+\alpha_{p,q}}\xi_j^{p,q} \tag{2.27}$$

with

$$\xi_j^{p,q} := \frac{\delta_j}{1+\delta_j L_j}\left(\sum_{l=j}^{q-1} w_l^{p,q}\frac{B_p-B_q}{B_{p,q}} + \frac{B_q}{B_{p,q}}\right).$$

2.3.2. Approximate affine swap rate dynamics

In order to approximate the swap rate process with a pure square-root volatility process we introduce the process

$$dv^{p,q} = \kappa^{p,q}(\theta^{p,q} - v^{p,q})dt + \sqrt{v^{p,q}}\left(\sigma_{p,q}^\top dW^{(n)} + \overline{\sigma}_{p,q}^\top d\overline{W}^{(n)}\right), \quad v^{p,q}(0) = \theta^{p,q} \tag{2.28}$$

with

$$\theta^{p,q} := \sum_{l=p}^{q-1} w_l^{p,q}(0)\theta_l \tag{2.29}$$
$$\kappa^{p,q} := \sum_{l=p}^{q-1} w_l^{p,q}(0)\kappa_l$$
$$\sigma_{p,q} := \sum_{l=p}^{q-1} w_l^{p,q}(0)\sigma_l$$
$$\overline{\sigma}_{p,q} := \sum_{l=p}^{q-1} w_l^{p,q}(0)\overline{\sigma}_l.$$

Define further

$$\alpha^{p,q} := \sum_{l=p}^{q-1} w_l^{p,q}(0)\alpha_l.$$

By replacing in (2.27) all volatility processes v_j with the, in a sense, averaged process $v^{p,q}$, and freezing Libors we arrive at the approximation

$$\begin{aligned}
\frac{S_{p,q}}{S_{p,q}+\alpha_{p,q}}\Lambda_{p,q} &\approx \sum_{j=p}^{q-1}\sum_{j=p}^{q-1}\begin{bmatrix}\sqrt{v^{p,q}}\beta_j \\ \\ \gamma_j\end{bmatrix}\left[\frac{L_j+\alpha_j}{S_{p,q}+\alpha_{p,q}}\xi_j^{p,q}\right](0) \\
&= \begin{bmatrix}\sqrt{v^{p,q}}\beta_{p,q} \\ \\ \gamma_{p,q}\end{bmatrix}, \quad \text{where} \\
\beta_{p,q} &:= \sum_{j=p}^{q-1}\beta_j\left[\frac{L_j+\alpha_j}{S_{p,q}+\alpha_{p,q}}\xi_j^{p,q}\right](0) \quad \text{and} \\
\gamma_{p,q} &:= \sum_{j=p}^{q-1}\gamma_j\left[\frac{L_j+\alpha_j}{S_{p,q}+\alpha_{p,q}}\xi_j^{p,q}\right](0)
\end{aligned}$$

(note that $\sum_{j=p}^{q-1}\xi_j^{p,q}\left(L_j+\alpha_j\right)/\left(S_{p,q}+\alpha_{p,q}\right)\approx 1$), hence yielding affine approximative swap rate dynamics

$$d\ln\left(S_{p,q}+\alpha_{p,q}\right) = -\frac{1}{2}v^{p,q}\left|\beta_{pq}\right|^2 dt - \frac{1}{2}\left|\gamma_{p,q}\right|^2 dt + \sqrt{v^{p,q}}\beta_{p,q}^\top dW^{p,q} + \gamma_{p,q}^\top d\widehat{W}^{p,q}. \quad (2.30)$$

For the (approximate) dynamics of $v^{p,q}$ under the annuity Brownian motions we replace in (2.24) the processes v_k by $v^{p,q}\left(1_{j<q}+\frac{\theta_j}{\theta_{p,q}}1_{j\geq q}\right)$, and freeze the Libors as usual. For $q=p+1$ this leads to the same approximation as for the corresponding caplet. From (2.28) we then obtain (as in Section 2.2, it follows again that $\overline{W}^{(n)}=\overline{W}^{p,q}$),

$$\begin{aligned}
dv^{p,q} &\approx \kappa^{p,q}(\theta^{p,q}-v^{p,q})dt + \sqrt{v^{p,q}}\overline{\sigma}_{p,q}^\top d\overline{W}^{(n)} + \sqrt{v^{p,q}}\sigma_{p,q}^\top \cdot \\
&\left(dW^{p,q} + \sqrt{v^{p,q}}dt\sum_{l=p}^{q-1}\sum_{k=l+1}^{n-1}\left[\sqrt{1_{k<q}+\frac{\theta_k}{\theta_{p,q}}1_{k\geq q}}w_l^{p,q}\frac{\delta_k\left(L_k+\alpha_k\right)}{1+\delta_k L_k}\right](0)\,\beta_k\right).
\end{aligned}$$

By setting

$$\widetilde{\kappa}^{p,q} := \kappa^{p,q} - \sum_{l=p}^{q-1}\left[w_l^{p,q}\sum_{k=l+1}^{n-1}\sqrt{1_{k<q}+\frac{\theta_k}{\theta_{p,q}}1_{k\geq q}}\frac{\delta_k\left(L_k+\alpha_k\right)}{1+\delta_k L_k}\right](0)\,\sigma_{p,q}^\top\beta_k \quad (2.31)$$

$$\widetilde{\theta}^{p,q} = \frac{\kappa^{p,q}\theta^{p,q}}{\widetilde{\kappa}^{p,q}}, \quad (2.32)$$

we thus have (in approximation)

$$dv^{p,q} = \widetilde{\kappa}^{p,q}(\widetilde{\theta}^{p,q}-v^{p,q})dt + \sqrt{v^{p,q}}\sigma_{p,q}^\top dW^{p,q} + \sqrt{v^{p,q}}\overline{\sigma}_{p,q}^\top d\overline{W}^{p,q}. \quad (2.33)$$

2.3.3. Fourier based swaption pricing

A (payer) swaption over the period $[T_p, T_q]$ is the option to enter at T_p into a swap over the period $[T_p, T_q]$ with strike K. It follows straightforwardly that the value at time $t = 0$ is given by

$$Swpn_{p,q}(K) = B_{p,q}(0)E_{p,q}\left[(S_{p,q}(T_p) - K)^+\right] \tag{2.34}$$

$$= B_{p,q}(0)E_{p,q}\left[\left((S_{p,q}(0) + \alpha_{p,q})e^{\ln\frac{S_{p,q}(T_p)+\alpha_{p,q}}{S_{p,q}(0)+\alpha_{p,q}}} - (K + \alpha_{p,q})\right)^+\right] \tag{2.35}$$

$$=: B_{p,q}(0)E_{p,q}\left[\left(S^{disp}_{p,q}(0)e^{\ln\frac{S^{disp}_{p,q}(T_p)}{S^{disp}_{p,q}(0)}} - K^{disp}_{p,q}\right)^+\right]. \tag{2.36}$$

Thus, after determining the characteristic function for $\ln\left[S^{disp}_{p,q}(T_p)/S^{disp}_{p,q}(0)\right]$ we may price the option by the Carr-Madan Fourier inversion method, just like we did for caplets in Subsection 2.2.1. Recalling the analysis from Subsection 2.2.1 it follows immediately that this characteristic function is given by

$$\varphi^{disp}_{p,q}(z\,;v) := E_{p,q}\left[e^{\mathrm{i}z\ln\frac{S^{disp}_{p,q}(T_p)}{S^{disp}_{p,q}(0)}}\,\middle|\, v_{p,q}(0) = v\right] \tag{2.37}$$

$$\exp\left(A_{p,q}(z;T_p) + B_{p,q}(z;T_p)v\right)\exp\left(-\frac{1}{2}\left(\mathrm{i}z + z^2\right)\int_0^{T_p}|\gamma_{p,q}|^2\,ds\right),$$

where

$$B_{p,q}(z;T_p) = \frac{a_{p,q} + d_{p,q}}{\varepsilon^2_{p,q}}\frac{1 - e^{d_{p,q}T_p}}{1 - g_{p,q}e^{d_{p,q}T_p}}$$

and

$$A_{p,q}(z;T) = \frac{\widetilde{\kappa}^{p,q}\widetilde{\theta}^{p,q}}{|\sigma_{p,q}|^2 + |\overline{\sigma}_{p,q}|^2}\left\{\left(a_{p,q} - d_{p,q}\right)T_p - 2\ln\left[\frac{e^{-d_{p,q}T_p} - g_{p,q}}{1 - g_{p,q}}\right]\right\}$$

with

$$\begin{aligned}
a_{p,q} &= \widetilde{\kappa}^{p,q} - \mathrm{i}z\sigma^\top_{p,q}\beta_{p.q} \\
d_{p,q} &= \sqrt{a^2_{p,q} + |\beta_{p,q}|^2\left(\mathrm{i}z + z^2\right)\left(|\sigma_{p,q}|^2 + |\overline{\sigma}_{p,q}|^2\right)} \\
g_{p,q} &= \frac{a_{p,q} + d_{p,q}}{a_{p,q} - d_{p,q}}.
\end{aligned}$$

Based on (2.37) the (approximate) price of a swaption with maturity T_p and swaption leg $[T_p, T_q]$ is given by

$$Swpn_{p,q}(K) = B_{p,q}(0)E_{p,q}\left[\left(S^{disp}_{p,q}(T_p) - K^{disp}_{p,q}\right)^+\right] \tag{2.38}$$
$$\approx Swpn^{\mathcal{B},disp}_{p,q}(K) +$$
$$\frac{B_{p,q}(0)S^{disp}_{p,q}(0)}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi^{\mathcal{B},disp}_{p,q}(z-\mathfrak{i};T_p,\theta_{p,q}) - \varphi^{disp}_{p,q}(z-\mathfrak{i};T_p,\theta_{p,q})}{z(z-\mathfrak{i})}e^{-\mathfrak{i}z\ln\frac{K^{disp}_{p,q}}{S^{disp}_{p,q}(0)}}dz$$

In (2.38), $\varphi^{\mathcal{B},disp}_{p,q}$ is the characteristic function of a corresponding Black model,

$$S^{disp}_{p,q}(T_p) = S^{disp}_{p,q}(0)e^{-\frac{1}{2}\left(\sigma^B_{p,q}\right)^2 T_p + \sigma^B_{p,q}\sqrt{T_p}\varsigma}, \quad \varsigma \in N(0,1),$$

where $\sigma^B_{p,q}$ is a suitably chosen volatility, and

$$\begin{aligned} Swpn^{\mathcal{B},disp}_{p,q}(K) &= B_{p,q}(0)E_{p,q}\left(S^{disp}_{p,q}(T_p) - K^{disp}_{p,q}\right)^+ \\ &= B_{p,q}(0)\mathcal{B}(S^{disp}_{p,q}(0), T_p, \sigma^B_{p,q}, K^{disp}_{p,q}), \end{aligned}$$

is given by Black's formula (cf. (2.18)).

2.3.4. Putting the swaption approximation to the test

In the same spirit as we have tested the caplet price approximation in Subsection 2.2.3 we now test the above Fourier based swaption pricing method. For each pair (p,q), $1 \le p < q \le n$ $(q \ne p+1)$, we replace all volatility processes v_j, $p \le j < q$, with $v_{p,q}$ given by (2.28), (2.29) to obtain in fact a Wu-Zhang related swaption approximation model linked to this pair (p,q). We then compare the simulated (p,q)-swaption price due to the "true" model (2.5) and the model with common stochastic volatility process (2.28). In turn, the latter price can be accurately approximated by (2.38) as shown in Wu and Zhang [2008]. We base the numerical experiments on the same data set as in Subsection 2.2.3.

In detail, this means that for putting up the "true" and the approximate Libor model, the initial Libors are stripped from a given spot rate curve and their values are given in Table 2.1, the Gaussian γ-part is deactivated by putting $\gamma_j \equiv 0$ and no displacement is in force by choosing $\alpha_j \equiv 0$. Moreover, the parametrization of the correlation structure from Subsection 2.2.4 is given by

$$r_{ij} = \exp\big(-0.118|T_i - T_j|\big) = e_i^\top e_j, \;\; \beta_j = 0.15e_j,$$

with the orthonormal vectors e_j resulting from a Cholesky decomposition of (r_{ij}) and $\delta_j = T_{j+1} - T_j \equiv 1.0$ and $\theta_j \equiv 1$ remain valid. All other simulation parameters, in particular the ρ_j's, κ_j's and ε_j's can be found in Table 2.1 and we retain the diffusion coefficients

$$\sigma_j = \rho_j \varepsilon_j e_j, \ \overline{\sigma}_j = \sqrt{1-\rho_j^2}\varepsilon_j.$$

To gear towards the approximate Libor model, we perform the calculation of the weighted volatility parameters $\kappa^{p,q}$, $\theta^{p,q}$, $\sigma^{p,q}$ and $\bar{\sigma}^{p,q}$ according to (2.29), where the frozen weights $w_l^{p,q}(0)$ are given in (2.23), so that the averaged approximate volatility process $v^{p,q}$ from (2.28) can be simulated. This averaged stochastic volatility is then reinserted into the Libor dynamics (2.3), i.e. $v^{p,q}$ virtually replaces each expiry-wise volatility $v_j, j = p, \ldots, q-1$. The simulations are carried out using $30,000$ Monte Carlo paths.

We calculate "true" and approximate swaption prices for the payer swaption depicted in (2.34) for various strike levels and swap legs $[T_p, T_q]$. The results of our numerical experiments are depicted in Table 2.3.

The simulation results show that for swaption pricing, the approximate Libor model under one weighted stochastic volatility $v^{p,q}$ gives a surprisingly good fit to the true model dynamics (2.3), (2.1). Depending on the swap legs, absolute price deviations are in the range of basis points (for swaption maturing in two and four years) and in the range of ten basis points (for maturity ten years). Recalling that the approximation is somewhat strong as each expiry-wise volatility process $v_j, j = p, \ldots, q-1$ is replaced by one weighted volatility process $v^{p,q}$, the numerical results reveal however that we get reasonably well behaved approximations to the "true" model. Similar to Subsection 2.2.3, ITM and ATM contracts have moderate relative errors. This deteriorates as the strike level increases, leading to very low swaption prices and hence endows relative errors intrinsically with instability. Indeed, for far OTM swaptions, the absolute error becomes indistinguishable from the Monte Carlo simulation error.

2.4. Advanced calibration

In this Section we want to give an intuition of the strength of our Libor model by showing calibration results involving non-trivial parameters α_j and γ_j. We consider here Libors over 20 yrs on a half year basis resulting in 39 Libors. Therefore we have $\delta_j = T_{j+1} - T_j = 0.5$. We perform a calibration as outlined in Subsection 2.2.3 where the bond prices and the resulting parameters can be found in Table 2.4.

For the caplet calibration we achieve an overall averaged error of 1.36%. In the following we are going to show some of the fits as plots of implied volatility. Therefor we group the Libor panel in 4 groups of ten Libors (except the last group only containing 9, respectively) and show the best fit of each group in Figure 2.2 and the worse in Figure 2.3.

Within our model we are also able to fit to swaption cubes giving us surprisingly good fits. The calibration procedure involves the approximate swaption pricing as described in Subsection 2.3.3 using the parameters form Table 2.4 obtained due to the caplet calibration. As the Fourier based caplet prices do not involve the correlation structure we are free to use this for the calibration procedure. As it turned out the correlation

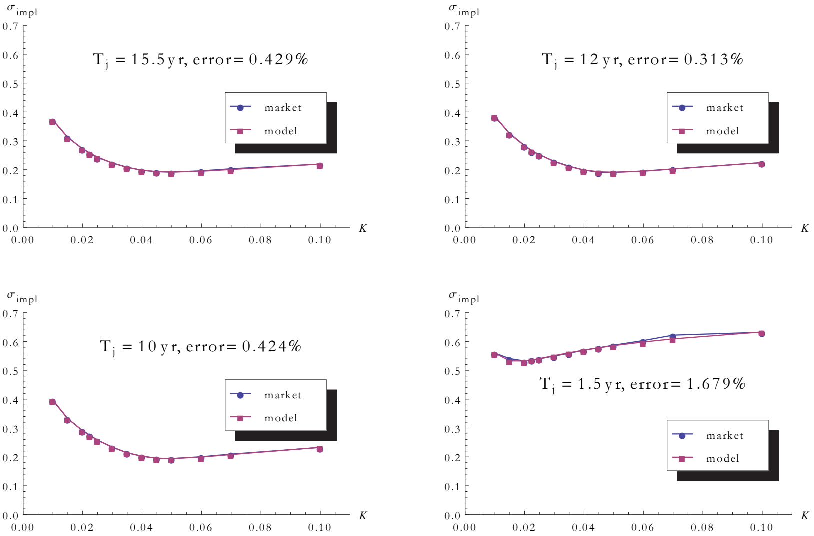

Figure 2.2.: Implied caplet volatilities due to market data vs. calibrated model - best fits

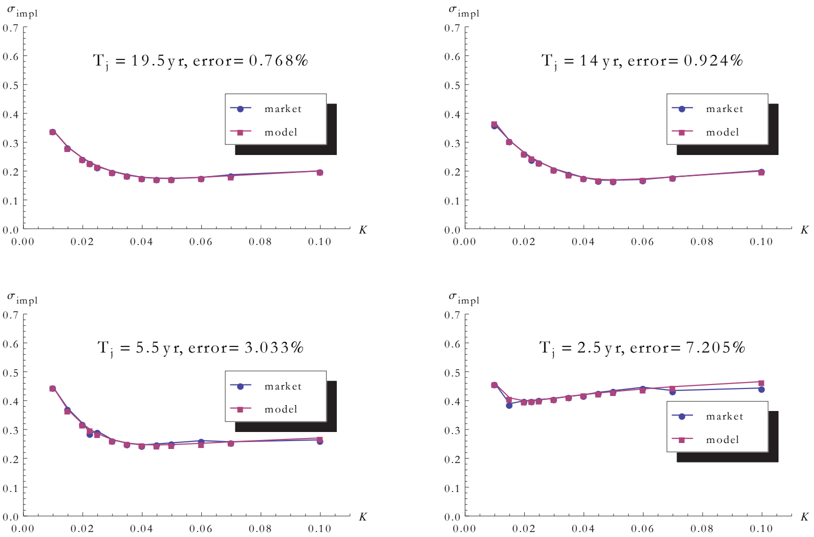

Figure 2.3.: Implied caplet volatilities due to market data vs. calibrated model - worse fits

structure used in Subsection 2.3.4 involving one parameter is not rich enough to obtain reasonable fits. We suggest to use the following correlation structure (cf. (14.19) in Andersen and Piterbarg [2010] (p.608)), involving 4 parameters, ρ_∞, a_0, a_∞ and κ.

$$Cor_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp\left(a\left(\min\left(i, j\right)\right) \delta \left|i - j\right|\right),$$

where $a(z) = a_\infty + (a_0 - a_\infty) \exp(-\kappa\delta z)$ subject to $0 \leq \rho_\infty \leq 1$, a_0, a_∞, $\kappa \geq 0$. This seems to be a good compromise between a good fit result and a fast calibration. The swaption cube involves strikes in the range of ± 150 *bp* the swap rate. The overall averaged error is given by 3.9%. For a visualization of our results we again choose implied volatility plots given in Figure 2.4.

2.5. Outlook

We established here an extended Libor model allowing the pricing of caplets and swaptions in the same model. The model is equipped with great flexibility to be able to fit even "extreme market" quotes. The pricing is done quasi-analytical as the use of FFT is enabled after an approximation, resulting in an affine structure. Comprehensive numerical studies show that the approximation error lies within basis points for caplets as for swaptions, therefore yielding a practically tractable pricing tool.

The main focus of future research will be placed on the incorporation of a multi-curve setting. As most Libor models presented in the literature so far this model is based on the assumption that Libors with shorter maturity can be used to replicate those with longer ones. For example an investment in a Libor with a maturity of 6 month should yield approximately the same gain as an investment in corresponding 3 month Libors with reinvestment after 3 month.

A further issue of concern is given by an efficient simulation of the Libor and swap rate dynamics. Some first ideas are presented in Chapter 3.

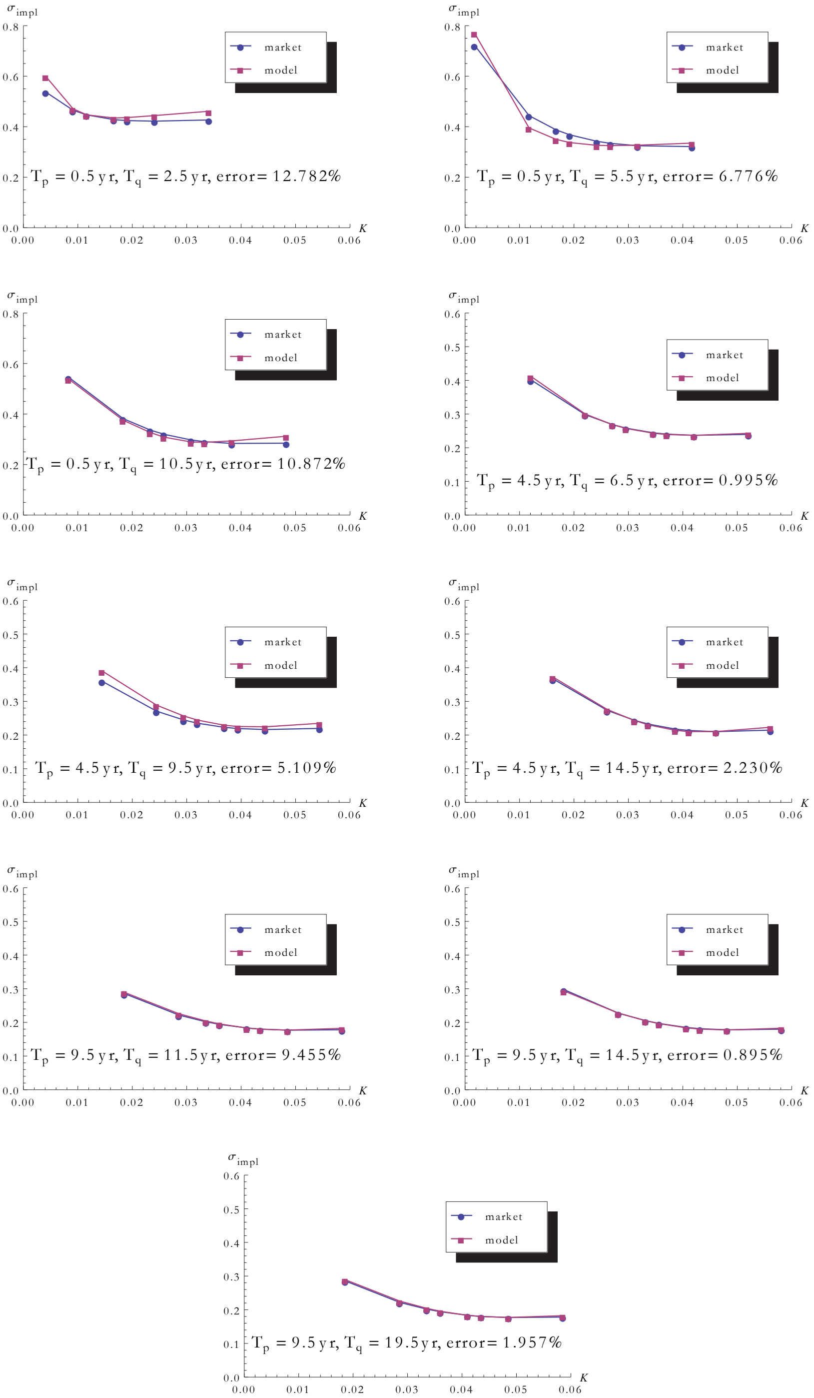

Figure 2.4.: Implied swaption volatilities due to market data vs. calibrated model

$[T_p, T_q]$	Strike	Price (SE)	Approx. price (SE)	Abs. error	Rel. error
[2, 10]	0.000	0.1640 (2.1e-04)	0.1637 (2.1e-04)	0.00032	0.002
	0.005	0.1302 (2.0e-04)	0.1299 (2.0e-04)	0.00032	0.002
	0.010	0.0964 (1.9e-04)	0.0961 (1.9e-04)	0.00031	0.003
	0.015	0.0628 (1.8e-04)	0.0625 (1.8e-04)	0.00033	0.005
	0.020	0.0317 (1.5e-04)	0.0313 (1.5e-04)	0.00037	0.011
	0.025	0.0094 (9.0e-05)	0.0092 (9.0e-04)	0.00024	0.026
	0.030	0.0011 (3.0e-05)	0.0010 (2.9e-05)	0.00003	0.030
	0.035	4.96e-05 (5.8e-06)	5.05e-05 (6.0e-06)	-9.33e-07	-0.01
	0.040	1.14e-06 (8.3e-07)	1.61e-06 (9.6e-07)	-4.71e-07	-0.41
[4, 10]	0.000	0.1228 (2.3e-04)	0.1223 (2.3e-04)	0.00057	0.004
	0.005	0.0981 (2.2e-04)	0.0975 (2.2e-04)	0.00055	0.005
	0.010	0.0734 (2.1e-04)	0.0728 (2.1e-04)	0.00055	0.007
	0.015	0.0493 (2.0e-04)	0.0488 (1.9e-04)	0.00057	0.011
	0.020	0.0281 (1.6e-04)	0.0275 (1.6e-04)	0.00060	0.021
	0.025	0.0127 (1.2e-04)	0.0122 (1.1e-04)	0.00049	0.038
	0.030	0.0042 (7.1e-05)	0.0040 (6.9e-05)	0.00026	0.060
	0.035	0.0010 (3.5e-05)	0.0009 (3.4e-05)	0.00008	0.076
	0.040	0.0002 (1.5e-05)	0.00019 (1.5e-05)	0.00001	0.071
[4, 20]	0.000	0.2877 (4.8e-04)	0.2866 (4.8e-04)	0.00110	0.003
	0.005	0.2288 (4.6e-04)	0.2277 (4.6e-04)	0.00107	0.004
	0.010	0.1699 (4.5e-04)	0.1689 (4.4e-04)	0.00104	0.006
	0.015	0.1122 (4.2e-04)	0.1112 (4.2e-04)	0.00102	0.009
	0.020	0.0609 (3.5e-04)	0.0600 (3.5e-04)	0.00091	0.015
	0.025	0.0246 (2.4e-04)	0.0241 (2.4e-04)	0.00051	0.020
	0.030	0.0068 (1.2e-04)	0.0067 (1.2e-04)	0.00011	0.016
	0.035	0.0012 (5.6e-05)	0.0013 (5.8e-05)	-0.00003	-0.023
	0.040	1.93e-04 (2.4e-05)	2.17e-04 (2.6e-05)	-0.00002	-0.124
[10, 20]	0.000	0.1653 (4.5e-04)	0.1638 (4.4e-04)	0.00149	0.009
	0.005	0.1311 (4.4e-04)	0.1297 (4.3e-04)	0.00146	0.011
	0.010	0.0976 (4.2e-04)	0.0961 (4.1e-04)	0.00146	0.015
	0.015	0.0670 (3.9e-04)	0.0655 (3.8e-04)	0.00147	0.021
	0.020	0.0423 (3.3e-04)	0.0410 (3.3e-04)	0.00137	0.032
	0.025	0.0247 (2.7e-04)	0.0236 (2.6e-04)	0.00137	0.045
	0.030	0.0134 (2.0e-04)	0.0126 (1.9e-04)	0.00081	0.060
	0.035	0.0068 (1.4e-04)	0.0063 (1.4e-04)	0.00050	0.073
	0.040	0.0032 (1.0e-04)	0.0029 (9.8e-05)	0.00050	0.088

Table 2.3.: Simulation results for payer swaptions.

j	ρ_j	κ_j	ε_j	α_j	$\lvert\beta_j\rvert$	θ_j	$\lvert\gamma_j\rvert$	$B_j(0)$
0.5	0.71	1.94	1.39	0.007	0.34	1.06	0.132	0.989
1	0.38	0.04	1.16	0.007	0.34	2.54	0.133	0.977
1.5	0.42	0.04	1.17	0.009	0.34	2.51	0.139	0.969
2	-0.09	0.03	1.25	0.006	0.34	3.01	0.037	0.961
2.5	-0.25	0.06	1.09	0.005	0.34	2.17	0.024	0.955
3	0.04	0.05	0.78	0.007	0.34	2.09	0.000	0.949
3.5	-0.15	0.04	0.84	0.006	0.34	2.23	0.013	0.939
4	0.44	0.16	1.51	0.030	0.34	1.71	0.094	0.928
4.5	0.23	0.03	1.15	0.030	0.34	3.17	0.075	0.915
5	0.06	0.03	1.07	0.030	0.34	3.02	0.062	0.903
5.5	-0.11	0.03	1.00	0.030	0.34	2.88	0.020	0.889
6	-0.33	0.09	2.04	0.003	0.34	2.55	0.007	0.876
6.5	-0.37	0.03	2.49	0.005	0.34	4.58	0.040	0.861
7	-0.35	0.05	2.34	0.005	0.34	3.66	0.002	0.847
7.5	-0.37	0.02	2.62	0.006	0.34	5.47	0.016	0.832
8	-0.36	0.04	2.47	0.006	0.34	4.12	0.000	0.818
8.5	-0.36	0.01	2.58	0.007	0.34	7.23	0.027	0.804
9	-0.35	0.02	2.47	0.007	0.34	5.42	0.003	0.789
9.5	-0.34	0.02	2.66	0.007	0.34	6.33	0.010	0.775
10	-0.38	0.03	2.48	0.006	0.34	3.96	0.001	0.761
10.5	-0.38	0.03	2.85	0.007	0.34	3.98	0.000	0.746
11	-0.40	0.03	3.00	0.008	0.34	4.12	0.000	0.732
11.5	-0.40	0.02	2.97	0.008	0.34	4.69	0.000	0.718
12	-0.36	0.04	2.55	0.008	0.34	3.11	0.000	0.705
12.5	-0.35	0.04	2.86	0.008	0.34	2.92	0.000	0.691
13	-0.39	0.04	2.88	0.009	0.34	2.90	0.001	0.678
13.5	-0.37	0.04	2.94	0.009	0.34	2.88	0.000	0.666
14	-0.44	0.04	2.96	0.010	0.34	2.53	0.003	0.653
14.5	-0.42	0.04	3.00	0.009	0.34	2.36	0.000	0.641
15	-0.35	0.06	2.69	0.010	0.34	1.86	0.000	0.630
15.5	-0.33	0.06	2.70	0.010	0.34	1.78	0.000	0.618
16	-0.35	0.07	2.78	0.010	0.34	1.66	0.000	0.607
16.5	-0.33	0.07	2.90	0.010	0.34	1.56	0.000	0.596
17	-0.36	0.08	2.95	0.011	0.34	1.45	0.000	0.586
17.5	-0.34	0.08	2.89	0.010	0.34	1.36	0.001	0.576
18	-0.38	0.08	2.91	0.011	0.34	1.26	0.005	0.566
18.5	-0.36	0.09	2.86	0.010	0.34	1.17	0.000	0.556
19	-0.39	0.09	2.86	0.011	0.34	1.08	0.000	0.547
19.5	-0.37	0.10	3.00	0.011	0.34	1.00	0.008	0.538

Table 2.4.: Parameters of the Libor model and present values, terminal bond $B_{20}(0) = 0.529$.

3. Simulation and related optimization ideas

For practical issue we deal with some ideas for complexity reduction in a multidimensional Heston model. We will consider a somehow simplified model compared to the one introduced in Chapter 2 to keep the presentation readable and focus on the main problems. However these results extend to a model like it was presented in Chapter 2. Further this is done with respect to our project partners from HSH Nordbank as this is one result obtained within this paid cooperation.

3.1. Simulation

For the simulation of the Libor system (2.6) together with (2.1) we choose the terminal measure $\mathbb{P}_n$ as the simulation measure. No matter if we want to simulate the full blown Libors or the approximated dynamics the simulation of the vola processes under this measure is the same as the approximation only affects the drift due to measure change.

$$dv_j = \kappa_j(\theta_j - v_j)dt + \sqrt{v_j}\left(\sigma_j^\top dW^{(n)} + \overline{\sigma}_j^\top d\overline{W}^{(n)}\right), \quad v_j(0) = \theta_j$$

For the Libor dynamics we first want to look at the case of our approximated dynamics, namely

$$\begin{aligned} d\ln\left(L_j + \alpha_j\right) = & -\frac{1}{2}\left|\gamma_j\right|^2 dt - \frac{1}{2}v_j\left|\beta_j\right|^2 dt \\ & - \beta_j^\top v_j \sum_{k=j+1}^{n-1}\sqrt{\frac{\theta_k}{\theta_j}}\frac{\delta_k(L_k(0)+\alpha_k)}{1+\delta_k L_k(0)}\beta_k dt \\ & - \gamma_j^\top \sum_{k=j+1}^{n-1}\frac{\delta_k(L_k(0)+\alpha_k)}{1+\delta_k L_k(0)}\gamma_j^\top\gamma_k dt \\ & + \sqrt{v_j}\beta_j^\top dW^{(n)} + \gamma_j^\top d\widehat{W}^{(n)}. \end{aligned} \tag{3.1}$$

Notice from (3.1) that we are going to simulate the displaced log dynamics. The price of a caplet over $[T_j, T_{j+1}]$ at time t is given by

$$\delta_j B_{j+1}(t)\,\mathbb{E}_{j+1}^{\mathcal{F}_t}\left[\left(L_j\left(T_j\right) - K\right)^+\right]. \tag{3.2}$$

We are now going to deal with the problem that the price is given as an expectation with respect to the measure $\mathbb{P}_{j+1}$ but we are simulating in the measure $\mathbb{P}_n$. This can be easily solved by measure change, namely $\mathbb{E}_{j+1}^{\mathcal{F}_t}\left[C\left(T_j\right)\right] = \mathbb{E}_n^{\mathcal{F}_t}\left[C\left(T_j\right)\frac{d\mathbb{P}_{j+1}}{d\mathbb{P}_n}\right] / \mathbb{E}_n^{\mathcal{F}_t}\left[\frac{d\mathbb{P}_{j+1}}{d\mathbb{P}_n}\right]$ by Theorem 15, where

$$\frac{d\mathbb{P}_{j+1}}{d\mathbb{P}_n}\left(s\right) = \frac{B_{j+1}\left(s\right)}{B_n\left(s\right)} = \prod_{k=j+1}^{n-1}\frac{B_k\left(s\right)}{B_{k+1}\left(s\right)} = \frac{B_n\left(0\right)}{B_{j+1}\left(0\right)}\prod_{k=j+1}^{n-1}\left(1+\delta_k L_k\left(s\right)\right).$$

So, with $\mathbb{E}_n^{\mathcal{F}_t}\left[\frac{d\mathbb{P}_{j+1}}{d\mathbb{P}_n}\right] = \frac{B_{j+1}(t)}{B_n(t)}$ (3.2) becomes

$$\delta_j B_n\left(t\right)\mathbb{E}_n^{\mathcal{F}_t}\left[\left(L_j\left(T_j\right)-K\right)^+\prod_{k=j+1}^{n-1}\left(1+\delta_k L_k\left(T_j\right)\right)\right].$$

This expectation can be approximated by Monte Carlo methods using the fact that the Brownian motions involved are Brownian motions with respect to the measure $\mathbb{P}_n$. Note that the simulation of the full blown dynamics work the same by considering (2.6) instead of (3.1).

Taking into account the huge amount of literature on simulating Heston processes there is basically only one question left open for the simulation of the log-Libor dynamics as given by (3.1). How to deal with the displacement factor? Further questions how to reduce simulation time will be discussed in the later Sections of this Chapter. As already mentioned we recommend to simulate the displaced log-Libor dynamics and correct for the displacement at terminal time. Our calibration procedure involves the condition $L_j\left(0\right)+\alpha_j > 0$ for all j. However when simulating undisplaced dynamics we are not able to ensure $L_j\left(t\right)+\alpha_j > 0$, $t > 0$. Thus in the log-dynamics one may face the situation of a negative displaced Libor as only the Libor process stays positive almost sure. To avoid such situations we recommend to simulate displaced Libor dynamics and correct for the deterministic displacement at terminal time.

The swaption simulation was already laid out in Subsection 2.3.4. We suggest a simulation under the terminal measure $\mathbb{P}_n$ leaving us with the open question of measure change to the annuity measure $\mathbb{P}_{p,q}$. It holds with Theorem 15

$$\frac{d\mathbb{P}_{p,q}}{d\mathbb{P}_n}\left(s\right) = \frac{B_{p,q}\left(s\right)}{B_n\left(s\right)} = \frac{\sum_{j=p}^{q-1}B_j\left(s\right)}{B_n\left(s\right)} = \sum_{j=p}^{q-1}\left(\prod_{k=j}^{n-1}\frac{B_k\left(s\right)}{B_{k+1}\left(s\right)}\right).$$

3.2. PCA - Principal Component Analysis

Considering model (2.1) we first notice that we are free to choose the dimension of the Brownian motions between 1 and the number of Libors. We are even in a situation where the dimensions of the particular Brownian motions can be chosen differently. A PCA of a correlation matrix $\left(C_{ij}\right)_{1\leq i,j\leq n-1}$ can be seen as a map to a correlation matrix of rank F, $1 \leq F \leq n-1$. With that the process is only driven by the first F factors, where one chooses those with the biggest influence. They are given by the F largest eigenvalues and their corresponding eigenvectors. Given such a matrix $\left(C_{ij}^F\right)_{1\leq i,j\leq n-1}$ of rank F there exists a matrix $A \in \mathbb{R}^{n-1 \times F}$ such that $AA^T = C^F$. Our aim is to find a good approximation AdZ to dW, where Z is given by F independent Brownian motions.

As C is a quadratic matrix we are able to write $Cv_i = \lambda_i v_i$ for eigenvalue λ_i and corresponding eigenvector v_i. Let us assume that the eigenvalues are unequal and further ordered in the sense that $\lambda_1 > \lambda_2 > ... > \lambda_{n-1} > 0$. Otherwise we find a permutation matrix P, such that $P^T C^F P$ has the desired ordering property. For the first $1 \leq F \leq n-1$ eigenvalues it holds

$$C^F v^F = v^F \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_F \end{pmatrix},$$

where $v^F := (v_1, ..., v_F)$. Note that $C^F v^F = Cv^F$ coincide for the first F eigenvectors. Since C^F is assumed to be symmetric and $\lambda_i \neq \lambda_j$ for $i \neq j$ the eigenvectors are orthogonal to each other. We therefore have

$$\begin{aligned} C^F &= v^F \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_F \end{pmatrix} \left(v^F\right)^T \\ &= v^F \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_F} \end{pmatrix} \left(v^F \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_F} \end{pmatrix}\right)^T. \end{aligned}$$

It is easy to see that

$$A = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_F} \end{pmatrix} v^F$$

is the desired matrix. All that is left is a renormalization of the vectors of $A^T = (a_1, ..., a_{n-1})$. The new unit vectors are then given by

$$\tilde{e}_i := \frac{a_i}{\|a_i\|}.$$

3.3. Joshi's trick

The most time consuming part of the simulation of our Libor dynamics is the calculation of the drift. There we have to deal with objects of the form

$$\sum_{k=j+1}^{n-1} f_j \frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} g_k c_{ik}^F \tag{3.3}$$

for some f_j, g_k involving a rank reduced correlation matrix C^F, obtained after a PCA as described in Section 3.2. The last expression involves state-dependent variables, so in a simulation the drift term has to be calculated at each Monte Carlo step. For each

Libor L_i, $1 \leq i \leq n-1$, we have to perform $n-1-i$ summations. Thus for a whole set of Libors we are faced with approximately $n^2/2$ summations and thus multiplications. Even after a PCA the drift calculation still has a complexity of order n^2. Using our PCA knowledge we can reformulate (3.3).

$$
\begin{aligned}
\sum_{k=j+1}^{n-1} f_j \frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} g_k c_{ik}^F &= \sum_{k=j+1}^{n-1} f_j \frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} g_k \tilde{e}_j^T \tilde{e}_k \\
&= \sum_{k=j+1}^{n-1} \sum_{i=1}^{F} f_j \frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} g_k \tilde{e}_{ji} \tilde{e}_{ki}
\end{aligned}
$$

A further reformulation leads to an expression where the inner sum is independent of j.

$$
\sum_{k=j+1}^{n-1} \sum_{i=1}^{F} f_j \frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} g_k \tilde{e}_{ji} \tilde{e}_{ki} = \sum_{i=1}^{F} f_j \tilde{e}_{ji} \sum_{k=j+1}^{n-1} \frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} g_k \tilde{e}_{ki}
$$

If we define

$$
s_{i,j} := \sum_{k=j+1}^{n-1} \frac{\delta_k (L_k + \alpha_k)}{1 + \delta_k L_k} g_k \tilde{e}_{ki},
$$

$s_{i,j}$ satisfies

$$
s_{i,j} = s_{i,j+1} + \frac{\delta_{j+1} (L_{j+1} + \alpha_{j+1})}{1 + \delta_{j+1} L_{j+1}} g_{j+1} \tilde{e}_{(j+1)i}.
$$

Further, if the terms $\frac{\delta_{j+1}(L_{j+1}+\alpha_{j+1})}{1+\delta_{j+1}L_{j+1}} g_{j+1}$ are precomputed all terms $s_{i,j}$ can be calculated performing only $(n-1)F$ multiplications. For the computation of

$$
\sum_{i=1}^{F} f_j \tilde{e}_{ji} s_{i,j}
$$

another $(n-1)F$ multiplications are called for thus leaving the order of the total drift computation at $(n-1)F$.

3.4. Discussion of the multidimensional Heston model

In the one-dimensional setting the Heston model can be represented in three equivalent ways. To simplify the notation we assume for the forward rate process zero drift and write it down in log-dynamics. The first representation describes the correlation between the forward and the vola process explicitly.

$$
\begin{aligned}
d \ln (L) &= -\frac{1}{2} v dt + \sqrt{v} dW_L && (3.4a) \\
dv &= \kappa (\theta - v) dt + \sigma \sqrt{v} dW_v && (3.4b) \\
d \langle L, v \rangle &= \rho dt, && (3.4c)
\end{aligned}
$$

where W_L and W_v are two one-dimensional and independent Brownian motions. It is not necessary for our discussion to assume one-dimensionality for the Brownian motions. This is done to keep the presentation as simple as possible. However, to reduce the number of equations we can incorporate (3.4c) either into (3.4a) or (3.4b). Doing so one has to take care of the correlations between the two remaining processes to get an equivalent representation. This results either in

$$d\ln(L) = -\frac{1}{2}v dt + \rho\sqrt{v} dW_v + \sqrt{1-\rho^2}\sqrt{v} dW \tag{3.5a}$$

$$dv = \kappa(\theta - v) dt + \sigma\sqrt{v} dW_v \tag{3.5b}$$

or in

$$d\ln(L) = -\frac{1}{2}v dt + \sqrt{v} dW_L \tag{3.6a}$$

$$dv = \kappa(\theta - v) dt + \sigma\left(\rho\sqrt{v} dW_L + \sqrt{1-\rho^2}\sqrt{v} dW\right), \tag{3.6b}$$

where W is an one-dimensional Brownian motion independent of W_L and W_v. The results presented in Chapter 2 are based on a multidimensional extension of model (3.6). However in the multidimensional case we are not able to give an equivalent representation in terms of one of the other two models. Except for special cases this would destroy the instantaneous correlations. Consider the following example where we assume two forwards L_1 and L_2 equipped with their own volatility process v_1, v_2 respectively. Further each tuple of processes comes along with its own set of parameters ρ_i, κ_i, θ_i and σ_i, $i = 1, 2$. Considering model (3.5) we have

$$\begin{aligned}\mathrm{Cor}_{L_1,L_2} &:= \frac{\frac{dL_1}{L_1}\cdot\frac{dL_2}{L_2}}{\sqrt{\frac{dL_1}{L_1}\cdot\frac{dL_1}{L_1}}\sqrt{\frac{dL_2}{L_2}\cdot\frac{dL_2}{L_2}}} = \frac{\left(\rho_1\rho_2 + \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}\right)\sqrt{v_1 v_2}}{\sqrt{v_1}\sqrt{v_2}} \\ &= \left(\rho_1\rho_2 + \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}\right),\end{aligned}$$

whereas in model (3.6) we get

$$\mathrm{Cor}_{L_1,L_2} := \frac{\frac{dL_1}{L_1}\cdot\frac{dL_2}{L_2}}{\sqrt{\frac{dL_1}{L_1}\cdot\frac{dL_1}{L_1}}\sqrt{\frac{dL_2}{L_2}\cdot\frac{dL_2}{L_2}}} = \frac{\sqrt{v_1 v_2}}{\sqrt{v_1}\sqrt{v_2}} = 1.$$

As a result the two correlations differ as long as $\rho_1 \neq \rho_2$ and $\rho_i \neq 0$, $i = 1, 2$. Obviously one needs to choose either one of the models in before. The extended type (3.6) Heston model in Chapter 2 was chosen for no deeper reason. Indeed many practically relevant additional results concerning the Heston model assume the independent Brownian motion W to be incorporated in the forward process. To mention two of them is Benhamou, Gobet and Miri [2010] who gave closed form plain vanilla option prices using Malliavin calculus and Andersen [2007] proposing a new simulation scheme based on a moment matching idea outperforming an adjusted Euler scheme regarding computational complexity. As we already have a quasi closed-form calibration procedure using FFT there is no big need to accelerate it using the idea of Benhamou, Gobet and Miri [2010]. Especially since this induces an additional approximation error. So the gain would be at least negligible. We therefore focus on the Andersen scheme saving a significant amount of computational time.

We will continue with model (3.6). Let us point out again that we are able to reformulate the Libor model presented in Chapter 2 similar to model (3.6) where the ideas of the paper remain similar and need only to be adjusted slightly. The quality of the calibration also remains the same.

3.5. A multidimensional Andersen scheme

In this Section we extend the work by Andersen [2007] to a multidimensional case. As in Andersen [2007] we do this for equities. We further allow for displaced asset dynamics as recommended by Andersen [2007]. The model examined is given by

$$\begin{aligned} d\ln\left(X_j+\alpha_j\right) &= -\frac{1}{2}v_j dt + \rho_j\sqrt{v_j}e_j^\top dW_v \\ &+ \sqrt{1-\rho_j^2}\sqrt{v_j}e_j^\top dW, \quad j=1,...,n \end{aligned} \tag{3.7}$$

$$\begin{gathered} dv_j = \kappa_j(\theta_j - v_j)dt + \varepsilon_j\sqrt{v_j}e_j^\top dW_v, \quad v_j(0)=\theta_j, \quad j=1,...,n \\ W_v \perp W, \quad W_v, W \in \mathbb{R}^n \end{gathered} \tag{3.8}$$

where we use as less structural assumptions as possible. The unit vectors e_j are given after a Cholesky decomposition of the correlation matrix $c = (c_{ij})_{1\leq i,j\leq n}$. Differently to the one-dimensional case we are not free to stick W into the vola process or state the correlation between the assets and the vola processes separately. Except for special cases this would lead to a change of the instantaneous correlations (cf. discussion in the previous Section).

Remark 30 *As we work in an equity setting rather than a Libor one we denote the asset process with X differently to the Libor case.*

Working in a multidimensional setting we are nevertheless able to use many results of Andersen [2007] due to the fact that $e_j^\top W$ is again a standard normal distributed random variable for e_j being a unit vector.

Proposition 31 *For two independent Brownian motions X and Y, $X \sim \mathcal{N}\left(\mu_X, \sigma_X^2\right)$, $Y \sim \mathcal{N}\left(\mu_Y, \sigma_Y^2\right)$ and $a, b \in \mathbb{R}$ it holds that $aX + bY$ is $\mathcal{N}\left(a\mu_X + b\mu_X, a^2\sigma_X^2 + b^2\sigma_Y^2\right)$ distributed. So we have for a n-dimensional unit vector u and a n-dimensional standard Brownian motion W that $u^T W \sim \mathcal{N}(0,1)$.*

The presence of the displacement α will not cause problems, too. For the simulation of the log-asset dynamics one simply computes the displaced dynamics and shifts back afterwards. Therefore this is more a computational then a theoretical problem.

From the instantaneous correlations given above it is clear that for plain vanilla options on a single asset the same results can be expected due to $Cor_{X_j,v_j} = \rho_j$. These kind of options are independent of the underlying correlation structure $\left(e_i^T e_j\right)_{1\leq i,j\leq n}$. One should point out that this only holds true in expectation, not trajectory-wise. We will come back on this fact later on.

3.5.1. Discretization of the asset process

In the spirit of Andersen [2007] we look at the system (3.7) in integrated form.

$$\begin{aligned} \ln\left(X_j\left(t+\Delta\right)+\alpha_j\right) &= \ln\left(X_j\left(t\right)+\alpha_j\right) - \int_t^{t+\Delta} \frac{1}{2} v_j\left(u\right) du \\ &+ \int_t^{t+\Delta} \rho_j \sqrt{v_j\left(u\right)} e_j^\top dW_v\left(u\right) \\ &+ \int_t^{t+\Delta} \sqrt{1-\rho_j^2}\sqrt{v_j\left(u\right)} e_j^\top dW\left(u\right), \quad j=1,...,n \end{aligned} \tag{3.9}$$

$$\begin{aligned} v_j\left(t+\Delta\right) &= v_j\left(t\right) + \int_t^{t+\Delta} \kappa_j(\theta_j - v_j\left(u\right))du \\ &+ \int_t^{t+\Delta} \varepsilon_j \sqrt{v_j\left(u\right)} e_j^\top dW_v\left(u\right), \quad j=1,...,n \end{aligned} \tag{3.10}$$

$$\int_t^{t+\Delta} \sqrt{v_j\left(u\right)} e_j^\top dW_v\left(u\right) = \varepsilon_j^{-1}\left(v_j\left(t+\Delta\right) - v_j\left(t\right) - \kappa_j\theta_j\Delta + \kappa_j \int_t^{t+\Delta} v_j\left(u\right) du\right) \tag{3.11}$$

Substituting (3.11) in (3.9) yields

$$\begin{aligned}
\ln\left(X_j\left(t+\Delta\right)+\alpha_j\right) &= \ln\left(X_j\left(t\right)+\alpha_j\right)-\frac{1}{2}\int\limits_t^{t+\Delta} v_j\left(u\right)du \\
&+ \sqrt{1-\rho_j^2}\int\limits_t^{t+\Delta}\sqrt{v_j\left(u\right)}e_j^\top dW\left(u\right) \\
&+ \rho_j\varepsilon_j^{-1}\left(v_j\left(t+\Delta\right)-v_j\left(t\right)-\kappa_j\theta_j\Delta+\kappa_j\int\limits_t^{t+\Delta} v_j\left(u\right)du\right).
\end{aligned}$$

The last equation looks rather troublesome for $\varepsilon_j \to 0$ but in fact by (3.10) we see its good nature as v_j gets deterministic for $\varepsilon_j = 0$. Since W is independent of W_v and so independent of v_j, $j = 1, ..., n$, $\int_t^{t+\Delta}\sqrt{v_j\left(u\right)}e_j^\top dW\left(u\right)$ is a Gaussian random variable with mean 0 and variance $\int_t^{t+\Delta} v_j\left(u\right)du$, conditional on $v_j\left(t\right)$ and $\int_t^{t+\Delta} v_j\left(u\right)du$. Further we have to deal with the time integral of the vola process which will be simply approximated by

$$\int_t^{t+\Delta} v_j\left(u\right)du \approx \Delta\left[\gamma_1 v_j\left(t\right)+\gamma_2 v_j\left(t+\Delta\right)\right].$$

We stick here to a setting similar to Euler discretization by setting $\gamma_1 = 1$ and $\gamma_2 = 0$. Independently of the choice of γ_1 and γ_2 we have the following discretization scheme for the process X_j.

$$\begin{aligned}
\ln\left(\widehat{X}_j\left(t+\Delta\right)+\alpha_j\right) &= \ln\left(\widehat{X}_j\left(t\right)+\alpha_j\right) \\
&+ \left(\kappa_j\rho_j\varepsilon_j^{-1}-\frac{1}{2}\right)\Delta\left[\gamma_1\widehat{v}_j\left(t\right)+\gamma_2\widehat{v}_j\left(t+\Delta\right)\right] \\
&+ \rho_j\varepsilon_j^{-1}\left(\widehat{v}_j\left(t+\Delta\right)-\widehat{v}_j\left(t\right)-\kappa_j\theta_j\Delta\right) \\
&+ \sqrt{\Delta}\sqrt{1-\rho_j^2}\sqrt{\gamma_1\widehat{v}_j\left(t\right)+\gamma_2\widehat{v}_j\left(t+\Delta\right)}e_j^T Z,
\end{aligned} \tag{3.12}$$

where Z is a n-dimensional vector of mutually independent standard Gaussian random variables and $\widehat{v}$ a discrete time approximation of the vola process. To make the presentation more compact we introduce constants $K_0^j, ..., K_4^j$.

$$\begin{aligned}
K_0^j &= -\rho_j\varepsilon_j^{-1}\kappa_j\theta_j\Delta, \quad K_1^j = \left(\kappa_j\rho_j\varepsilon_j^{-1}-\frac{1}{2}\right)\Delta\gamma_1-\rho_j\varepsilon_j^{-1}, \\
K_3^j &= \left(\kappa_j\rho_j\varepsilon_j^{-1}-\frac{1}{2}\right)\Delta\gamma_2+\rho_j\varepsilon_j^{-1}, \\
K_4^j &= \Delta\left(1-\rho_j^2\right)\gamma_1, \quad K_5^j = \Delta\left(1-\rho_j^2\right)\gamma_2
\end{aligned} \tag{3.13}$$

Together with (3.12) this leads to

$$\begin{aligned}\ln\left(\widehat{X}_j(t+\Delta)+\alpha_j\right) &= \ln\left(\widehat{X}_j(t)+\alpha_j\right)+K_0^j+K_1^j\widehat{v}_j(t) \\ &\quad +K_2^j\widehat{v}_j(t+\Delta)+\sqrt{K_3^j\widehat{v}_j(t)+K_4^j\widehat{v}_j(t+\Delta)}e_j^TZ.\end{aligned} \tag{3.14}$$

3.5.2. Discretization schemes for the vola process

Following next we are going to concentrate on the simulation of the vola process. Let us therefore recall some important results concerning the vola processes v_j.

Proposition 32 *Let the cumulative distribution function for the non-central chi-square distribution with ν degrees of freedom and non-centrality parameter λ be given by*

$$F_{\chi'^2}(y;\nu,\lambda)=e^{-\lambda/2}\sum_{l=0}^{\infty}\frac{(\lambda/2)^l}{l!2^{\nu/2+j}\Gamma(\nu/2+l)}\int_0^z z^{\nu/2+l-1}e^{-x/2}dx.$$

Define

$$d:=4\kappa_j\theta_j/\epsilon_j^2,\quad n_j(t,T):=\frac{4\kappa_j e^{-\kappa_j(T-t)}}{\epsilon_j^2\left(1-e^{-\kappa_j(T-t)}\right)}.$$

For $T>t$, $v_j(T)$ as in (3.8), conditional on $v_j(t)$, is distributed as $e^{-\kappa_j(T-t)}/n_j(t,T)$ times a non-central chi-square distribution with d degrees of freedom and non-centrality parameter $v_j(t)\,n_j(t,T)$. This means the conditional probability is given by

$$\mathbb{P}\left[v_j(T)<x\,|v_j(t)\right]=F_{\chi'^2}\left(\frac{xn_j(t,T)}{e^{-\kappa_j(T-t)}};d,v_j(t)\,n_j(t,T)\right).$$

This Proposition follows by the standard literature, e. g. Cox, Ingersoll and Ross [1985].

For the non-central chi-square distribution the first two moments are well known and the results are stated in the following Corollary which follows immediately with Remark 31 since $e_j^\top dW_v$ is again a standard Brownian motion.

Corollary 33 *For $T>t$, conditional on $v_j(t)$, the following holds for $v_j(T)$.*

$$\begin{aligned}\mathbb{E}\left[v_j(T)\,|v_j(t)\right] &= \theta_j+(v_j(t)-\theta_j)\,e^{-\kappa_j(T-t)} \\ Var\left[v_j(T)\,|v_j(t)\right] &= \frac{v_j(t)\,\epsilon_j^2e^{-\kappa_j(T-t)}}{\kappa_j}\left(1-e^{-\kappa_j(T-t)}\right) \\ &\quad +\frac{\theta_j\epsilon_j^2}{2\kappa_j}\left(1-e^{-\kappa_j(T-t)}\right)^2\end{aligned}$$

3.5.3. TG scheme

In the spirit of Andersen [2007] we are going to develop the TG scheme by moment matching the vola approximation process

$$\widehat{v}_j\left(t+\Delta\right) := (\mu_j + \sigma_j e_j^T W)^+. \tag{3.15}$$

For calculation of μ_j and σ_j we will moment match the first two moments of the approximation, meaning $\mathbb{E}\left[\widehat{v}_j\left(t+\Delta\right)\right]$ and $\mathbb{E}\left[\widehat{v}_j\left(t+\Delta\right)^2\right]$ to the exact values given by $\mathbb{E}\left[v_j\left(t+\Delta\right)|v_j\left(t\right)=\widehat{v}_j\left(t\right)\right]$ and $\mathbb{E}\left[v_j\left(t+\Delta\right)^2|v_j\left(t\right)=\widehat{v}_j\left(t\right)\right]$. The result of the moment matching procedure is given in the following Proposition.

Proposition 34 *Let $\phi\left(x\right)=\left(2\pi\right)^{-1/2}e^{-x^2/2}$ be the standard Gaussian density and $\Phi\left(x\right)$ its cumulative distribution function. Then for $j=1,...,n$ define the functions $r_j:\mathbb{R}\to\mathbb{R}$ by*

$$r_j\left(x\right)\phi\left(r_j\left(x\right)\right)+\Phi\left(r_j\left(x\right)\right)\left(1+r_j\left(x\right)^2\right)=\left(1+x\right)\left(\phi r_j\left(\left(x\right)\right)+r_j\left(x\right)\Phi\left(r_j\left(x\right)\right)\right).$$

Also define

$$\begin{aligned} m_j &:= \mathbb{E}\left[\widehat{v}_j\left(t+\Delta\right)|v_j\left(t\right)=\widehat{v}_j\left(t\right)\right], && (3.16a)\\ s_j^2 &:= Var\left[\widehat{v}_j\left(t+\Delta\right)|v_j\left(t\right)=\widehat{v}_j\left(t\right)\right], && (3.16b)\\ \psi_j &:= s_j^2/m_j^2. && (3.16c)\end{aligned}$$

Generating $\widehat{v}_j\left(t+\Delta\right)$ with the TG scheme together with the parameter setting

$$\mu_j=\frac{m_j}{\Phi(\mu_j/\sigma_j)+(\sigma_j/\mu_j)\,\phi(\mu_j/\sigma_j)},\quad \sigma_j=\frac{m_j}{(\mu_j/\sigma_j)\Phi(\mu_j/\sigma_j)+\phi(\mu_j/\sigma_j)}$$

results in $\mathbb{E}\left[\widehat{v}_j\left(t+\Delta\right)\right]=m_j$ and $Var\left[\widehat{v}_j\left(t+\Delta\right)\right]=s_j^2$.

The proof follows immediately with Andersen [2007] and Remark 31. A recovery of the functions r_j must be done by numerical root-search due to the non-linearity of the equations involved. As recommended in Andersen [2007] it seems to be advisable to map out functions $r_j=r_j\left(\psi_j\right)$ once on an equidistant grid and due an easy look-up every time $r_j\left(\psi_j\right)$ is needed. One can show

$$\psi_j\in\left(0,\epsilon_j^2/\left(2\kappa_j\theta_j\right)\right].$$

Up to know we did not say a word about the correlations between different vola processes and whether the correlation structure is preserved by approximation (3.15) or not. Numerical examples suggest that the approximation works well. Nevertheless we will deal with a moment matching procedure for the correlations which is postponed to Section 3.6.

3.5.4. QE scheme

Next let us introduce the QE scheme, a quadratic approximation of the vola process which guarantees non-negativity. The QE scheme does not need pre-caching due to the fact, that the parameters involved can be calculated explicitly. This is another advantage compared to the TG scheme. For the approximation we choose

$$\widehat{v}_j(t+\Delta) := a_j \left(b_j + e_j^T W\right)^2. \tag{3.17}$$

Unfortunately this approximation only yields good results for sufficiently large values of $\widehat{v}_j(t)$. For low values we will approximate the asymptotic density by

$$\mathbb{P}\left[\widehat{v}_j(t+\Delta) \in [x, x+dx]\right] \approx \left(p_j \delta(0) + \beta_j (1-p_j) e^{-\beta_j x}\right) dx. \tag{3.18}$$

$\delta(\cdot)$ here denotes the Dirac function and p and β are non-negative parameters yet to be determined via moment matching. This approach is inspired by the fact that for low values of v_j the vola process behaves similar to a chi-squared distribution, as the non-centrality parameter approaches zero. The distribution function of a chi-square random variable with ν degrees of freedom is given by

$$f_{\chi^2}(x;\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-x^2/2} x^{\nu/2-1}.$$

By integrating equation (3.18) we get the cumulative distribution function

$$\psi_j(x) := \mathbb{P}\left[\widehat{v}_j(t+\Delta) \le x\right] = p_j + (1-p_j)\left(1 - e^{-\beta_j x}\right), \quad x \ge 0.$$

Using the standard inverse distribution function method we notice

$$\widehat{v}_j(t+\Delta) = \psi_j^{-1}\left(\Phi\left(e_j^T W_v\right); p_j, \beta_j\right) \tag{3.19}$$

where the inverse function can be computed by

$$\psi_j^{-1}(u; p, \beta) = \begin{cases} 0, & 0 \le u \le p \\ \beta^{-1} \ln\left(\frac{1-p}{1-u}\right), & p < u \le 1. \end{cases} \tag{3.20}$$

The calculation of p_j and β_j is stated in the next Proposition.

Proposition 35 *Let m_j, s_j and ψ_j denote the same quantities as in Proposition (34). For $\psi_j \ge 1$, set*

$$p_j = \frac{\psi_j - 1}{\psi_j + 1}$$

and

$$\beta_j = \frac{1-p_j}{m_j}.$$

One has for $\widehat{v}_j(t+\Delta)$, as defined in (3.19), $\mathbb{E}[\widehat{v}_j(t+\Delta)] = m_j$ and $Var[\widehat{v}_j(t+\Delta)] = s_j^2$.

For high values of $\widehat{v}_j(t)$ the approximation is given by (3.17) where the computation of a_j and b_j is given in the following Proposition.

Proposition 36 *Let again m_j, s_j and ψ_j be given as in Proposition (34). Given $\psi_j \leq 2$ set*

$$b_j^2 = 2\psi_j^{-1} - 1 + \sqrt{2\psi_j^{-1}}\sqrt{2\psi_j^{-1} - 1}$$

and

$$a_j = \frac{m_j}{1+b_j^2}.$$

For $\widehat{v}_j(t+\Delta)$ as defined in (3.17) one has $\mathbb{E}[\widehat{v}_j(t+\Delta)] = m_j$ and $Var[\widehat{v}_j(t+\Delta)] = s_j^2$.

Let us shortly discuss the first two moments of approximation (3.17). One has

$$\begin{aligned}
\mathbb{E}[\widehat{v}_j(t+\Delta)] &= \mathbb{E}\left[a_j\left(b_j + e_j^T W\right)^2\right] \\
&= \mathbb{E}\left[a_j\left(b_j^2 + 2b_j e_j^T W + \left(e_j^T W\right)^2\right)\right] \\
&= a_j\left(b_j^2 + 1\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\widehat{v}_j(t+\Delta)^2\right] &= \mathbb{E}\left[a_j^2\left(b_j + e_j^T W\right)^4\right] \\
&= a_j^2 \mathbb{E}\left[b_j^4 + 4b_j^3 e_j^T W + 6b_j^2\left(e_j^T W\right)^2 + 4b_j\left(e_j^T W\right)^3 + \left(e_j^T W\right)^4\right] \\
&= a_j^2\left(b_j^4 + 6b_j^2 + \mathbb{E}\left[\left(e_j^T W\right)^4\right]\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[\left(e_j^T W\right)^4\right] &= \mathbb{E}\left[\sum_i\left(e_{ji}^4 W_i^4\right) + 6\sum_{k,l,k\neq l}\left(e_{jk}^2 e_{jl}^2 W_k^2 W_l^2\right)\right] \\
&= 3\sum_i e_{ji}^4 + 6\sum_{k,l,k\neq l} e_{jk}^2 e_{jl}^2 \\
&= 3\left(\sum_i e_{ji}^2\right)^2 \\
&= 3.
\end{aligned}$$

Above, all odd terms with respect to the Brownian motion vanish and we have

$$\mathbb{E}\left[\widehat{v}_j(t+\Delta)^2\right] = a_j^2\left(b_j^4 + 6b_j^2 + 3\right),$$

thus

$$\begin{aligned}\mathbb{E}\left[\widehat{v}_j(t+\Delta)^2\right]-\mathbb{E}\left[\widehat{v}_j(t+\Delta)\right]^2 &= a_j^2\left(b_j^4+6b_j^2+3\right)-\left(a_j\left(b_j^2+1\right)\right)^2\\ &= a_j^2\left(b_j^4+6b_j^2+3\right)-a_j^2\left(b_j^4+2b_j^2+1\right)\\ &= 2a_j^2\left(1+2b_j^2\right).\end{aligned}$$

3.5.5. Martingale correction

Equivalently for (3.14) we may write

$$\begin{aligned}\widehat{X}_j(t+\Delta)+\alpha_j &= \left(\widehat{X}_j(t)+\alpha_j\right)\exp\left(K_0^j+K_1^j\widehat{v}_j(t)\right)\cdot\\ &\quad\cdot\exp\left(K_2^j\widehat{v}_j(t+\Delta)+\sqrt{\left[K_3^j\widehat{v}_j(t)+K_4^j\widehat{v}_j(t+\Delta)\right]}e_j^TZ\right).\end{aligned}$$

For the corresponding continuous-time process X it can not be guaranteed that there exist finite higher order moments but it will always be a martingale, meaning

$$\mathbb{E}\left[X_j(t+\Delta)\,|X_j(t)\right]=X_j(t)<\infty.$$

A full-truncated Euler scheme (cf. Lord, Koekkoek and van Dijk [2010]) fulfills this property, too. As stated right now our approximation scheme (3.14) does not fulfill this property so we like to extend it in such a direction. A modification is called for such that

$$\mathbb{E}\left[\widehat{X}_j(t+\Delta)\middle|\widehat{X}_j(t)\right]=\widehat{X}_j(t).$$

As a by-product we will obtain sufficient conditions for $\mathbb{E}\left[\widehat{X}_j(t+\Delta)\middle|\widehat{X}_j(t)\right]$ to be bounded.

Proposition 37 *For K_i^j, $i=1,...,4$ as defined in (3.13) define*

$$M_j:=\mathbb{E}\left[\exp\left(A_j\widehat{v}_j(t+\Delta)\right)|\widehat{v}_j(t)\right]>0,\quad A_j:=K_2^j+\frac{1}{2}K_4^j.$$

For $M<\infty$ it follows that $\mathbb{E}\left[\widehat{X}_j(t+\Delta)\middle|\widehat{X}_j(t)\right]<\infty$. Assuming $M<\infty$ define

$$K_0^{j,\star}:=-\ln\left(M_j\right)-\left(K_1^j+\frac{1}{2}K_3^j\right)\widehat{v}_j(t). \tag{3.21}$$

Further for the approximation of the asset process set

$$\begin{aligned}\ln\left(\widehat{X}_j(t+\Delta)+\alpha_j\right) &= \ln\left(\widehat{X}_j(t)+\alpha_j\right)+K_0^{j,\star}+K_1^j\widehat{v}_j(t)+K_2^j\widehat{v}_j(t+\Delta) \quad (3.22)\\ &\quad+\sqrt{\left[K_3^j\widehat{v}_j(t)+K_4^j\widehat{v}_j(t+\Delta)\right]}e_j^TZ,\end{aligned}$$

with Z being a n-dimensional vector of mutually independent standard Gaussian random variables. We then have $\mathbb{E}\left[\widehat{X}_j(t+\Delta)\middle|\widehat{X}_j(t)\right] = \widehat{X}_j(t)$.

Proof. Looking at (3.22) in exponential form we have

$$\begin{aligned}
& \mathbb{E}\left[\widehat{X}_j(t+\Delta)+\alpha_j\middle|\widehat{X}_j(t),\widehat{v}_j(t)\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\widehat{X}_j(t+\Delta)+\alpha_j\middle|\widehat{X}_j(t),\widehat{v}_j(t),\widehat{v}_j(t+\Delta)\right]\right] \\
= \; & \left(\widehat{X}_j(t)+\alpha_j\right)\exp\left(K_0^{j,\star}+K_1^j\widehat{v}_j(t)\right)\mathbb{E}\left[\exp\left(K_2^j\widehat{v}_j(t+\Delta)\right)\cdot\right. \\
& \left.\cdot\,\mathbb{E}\left[\exp\left(\sqrt{\left[K_3^j\widehat{v}_j(t)+K_4^j\widehat{v}_j(t+\Delta)\right]}e_j^TZ\right)\middle|\widehat{X}_j(t),\widehat{v}_j(t),\widehat{v}_j(t+\Delta)\right]\right] \\
= \; & \left(\widehat{X}_j(t)+\alpha_j\right)\exp\left(K_0^{j,\star}+\left(K_1^j+\frac{1}{2}K_3^j\right)\widehat{v}_j(t)\right)\cdot \\
& \cdot\mathbb{E}\left[\exp\left(\left(K_2^j+\frac{1}{2}K_4^j\right)\widehat{v}_j(t+\Delta)\right)|\widehat{v}_j(t)\right] \\
= \; & \left(\widehat{X}_j(t)+\alpha_j\right)\exp\left(K_0^{j,\star}+\left(K_1^j+\frac{1}{2}K_3^j\right)\widehat{v}_j(t)\right)\mathbb{E}\left[\exp\left(A\widehat{v}_j(t+\Delta)\right)|\widehat{v}_j(t)\right].
\end{aligned}$$

From the last equation it is clear that we need

$$\exp\left(K_0^{j,\star}+\left(K_1^j+\frac{1}{2}K_3^j\right)\widehat{v}_j(t)\right)M_j = 1$$

to obtain the martingale property which yields (3.21). ■

The martingale correction only requires substitution of K_0^j by $K_0^{j,\star}$. Indeed this is only applicable in the case $M_j = \mathbb{E}\left[\exp\left(A_j\widehat{v}_j(t+\Delta)\right)|\widehat{v}_j(t)\right] < \infty$. We will focus on the case $\widehat{v}_j(t+\Delta) \geq 0$ as this involves both, TG and QE scheme. Then a sufficient condition for finiteness of M_j would be $A_j \leq 0$ which is the case for $\rho_j \leq 0$. This indeed is a very rough estimation. We will be able to give more precise results, also for the case $\rho_j > 0$, for a given approximation scheme of the vola process.

TG scheme

Proposition 38 *Let* $\widehat{v}_j(t+\Delta) = \left(\mu_j+\sigma_je_j^TW\right)^+$ *be given by the TG scheme. With*

$$d_j^+ = \frac{\mu_j}{\sigma_j}+A_j\sigma_j, \quad d_j^- = \frac{\mu_j}{\sigma_j}$$

we have for any value of A_j

$$\mathbb{E}\left[\exp\left(A_j\widehat{v}_j(t+\Delta)\right)|\widehat{v}_j(t)\right] = \exp\left(A_j\mu_j+\frac{1}{2}A_j^2\sigma_j^2\right)\Phi\left(d_j^+\right)+\Phi\left(d_j^-\right). \tag{3.23}$$

Proof. We have $\widehat{v}_j(t+\Delta) = \left(\mu_j + \sigma_j e_j^T W\right)^+$ where μ_j and σ_j depend on $\widehat{v}_j(t)$. It holds

$$\begin{aligned}
\mathbb{E}\left[\exp\left(A_j \widehat{v}_j(t+\Delta)\right) | \widehat{v}_j(t)\right] &= 1 + \mathbb{E}\left[\max\left\{\exp\left(A_j\mu_j + A_j\sigma_j e_j^T W\right) - 1, 0\right\}\right], \quad A \geq 0 \\
\mathbb{E}\left[\exp\left(A_j \widehat{v}_j(t+\Delta)\right) | \widehat{v}_j(t)\right] &= 1 - \mathbb{E}\left[\max\left\{1 - \exp\left(A_j\mu_j + A_j\sigma_j e_j^T W\right), 0\right\}\right], \quad A < 0.
\end{aligned}$$

With convolution it follows that $e_j^T W$ is standard normal distributed, thus we have that $\exp\left(A_j \widehat{v}_j(t+\Delta)\right)$ is log-normal. With standard results for the expectation of a log-normal variable and a Black-Scholes like call option analysis it follows (3.23). ■

QE scheme

Proposition 39 *Let $\psi_j^c \in [1,2]$ and $\psi_j = s_j^2/m_j^2$ be given, with m_j and s_j defined as in (3.16c). For $\psi_j \leq \psi_j^c$ let $\widehat{v}_j(t+\Delta) = a_j\left(b_j + e_j^T W\right)^2$, then*

$$\mathbb{E}\left[\exp\left(A_j \widehat{v}_j(t+\Delta)\right) | \widehat{v}_j(t)\right] = \frac{\exp\left(\frac{A_j a_j b_j^2}{1-2A_j a_j}\right)}{\sqrt{1-2A_j a_j}} \tag{3.24}$$

where

$$A_j < \frac{1}{2a_j} \tag{3.25}$$

is required for existence of a solution. Let $\widehat{v}_j(t+\Delta) = \psi_j^{-1}\left(\Phi\left(e_j^T W_v\right); p_j, \beta_j\right)$ for $\psi_j > \psi_j^c$ with ψ_j^{-1} as given in (3.20). Then

$$\mathbb{E}\left[\exp\left(A_j \widehat{v}_j(t+\Delta)\right) | \widehat{v}_j(t)\right] = p_j + \frac{\beta_j(1-p_j)}{\beta_j - A_j} \tag{3.26}$$

where

$$A_j < \beta_j \tag{3.27}$$

is required.

Proof. For $\psi_j \leq \psi_j^c$ it follows by standard arguments and the usual convolution Remark 31 that we have (3.24) where we need $1 - 2A_j a_j > 0$ to obtain a solution in $\mathbb{R}$, thus (3.25). For $\psi_j > \psi_j^c$ and direct integration we get

$$\begin{aligned}
\mathbb{E}\left[\exp\left(A_j \widehat{v}_j(t+\Delta)\right) | \widehat{v}_j(t)\right] &= \int_0^\infty \left(p_j\delta(0) + \beta_j(1-p_j)e^{-\beta_j x}\right)e^{A_j x}dx \\
&= p_j + \int_0^\infty \beta_j(1-p_j)e^{(A_j-\beta_j)x}dx \\
&= p_j + \frac{\beta_j(1-p_j)}{\beta_j - A_j},
\end{aligned}$$

assuming $A_j < \beta_j$. ■

Remark 40 *Note that there is a difference in result (3.26) compared to its equivalent in Andersen [2007] due to an error in Andersen [2007].*

3.5.6. Numerical example

The approximation schemes proposed by Andersen for the one dimensional case are based on a moment matching idea for the first two moments. For the multidimensional case this would mean one has to take into account the covariances, too. Intensive numerical experiments show, that the covariances are already fitted quite well, thus we won't consider a matching of the correlations here. Nevertheless a short discussion on that point of concern is given in Subsection 3.6.

Let us consider a correlation matrix $(c_{ij})_{1\leq i,j\leq 5}$ given by $c_{ij} = \exp(-\beta|i-j|)$. After a Cholesky decomposition we obtain unit vectors $(e_i)_{1\leq i\leq 5}$. Further we are given SDE's

$$\frac{dX_i(t)}{X_i(t)+\alpha_i} = \mu_i(t)\,dt + \sigma_i e_i^T dW(t), \quad 1 \leq i \leq 5,$$

where W is a 5-dimensional vector of mutually independent Brownian motions. The drift is given by $\mu_i = \delta_i - r$, with δ_i being the continuous dividend payments of asset i and r being the discount factor. We assume here the existence of such a discount factor as discussed at the end of Section 1.1. σ_i is the volatility of asset i and α_i the displacement factor. We write X for $(X_i)_{1\leq i\leq 5}$ in short-hand notation. Further we consider here a European basket put to strike K with payoff structure at terminal time T, given by

$$H(X) := e^{-rT}\left(K - \frac{1}{5}\sum_{i=1}^{5} X_i(T)\right)^{+}.$$

Our simulation results are based on 10^6 trajectories by looking at the following parameter setting.

$$r = 0, \delta = 0, S_i(0) = 100, v_i(0) = \theta_i, T = 10, \alpha_i = 0, \beta = 0.3 \quad 1 \leq i \leq 5$$

	1	2	3	4	5
κ	0.1	1	1	1	1
ϵ	0.5	0.5	1.5	0.5	1.5
θ	0.1	0.1	0.1	0.04	0.04
ρ	-0.9	-0.9	-0.9	-0.9	-0.9

Table 3.1.: Model parameters

The simulation results can be found in Table 3.2 showing the option price and the sample standard deviations in parentheses.

$\rho_{ij} = \exp(-0.3|i-j|)$

Δ	Euler	TG	TG mart	QE	QE mart
			K=100		
1	14.091(0.021)	10.186(0.017)	10.168(0.017)	10.223(0.017)	10.075(0.016)
1/2	12.067(0.019)	9.991(0.016)	10.039(0.016)	10.072(0.016)	9.982(0.016)
1/4	10.765(0.017)	9.881(0.016)	9.961(0.016)	9.968(0.016)	9.930(0.016)
1/8	10.106(0.016)	9.808(0.016)	9.881(0.016)	9.899(0.016)	9.886(0.016)
1/16	9.846(0.016)	9.747(0.016)	9.797(0.016)	9.837(0.016)	9.833(0.016)
1/32	9.740(0.016)	9.713(0.016)	9.715(0.016)	9.805(0.016)	9.804(0.016)
			K=70		
1	4.023(0.010)	2.182(0.008)	2.092(0.007)	2.197(0.008)	2.071(0.007)
1/2	3.063(0.009)	2.019(0.007)	2.005(0.007)	2.048(0.007)	1.990(0.007)
1/4	2.394(0.008)	1.934(0.007)	1.951(0.007)	1.974(0.007)	1.953(0.007)
1/8	2.045(0.007)	1.882(0.007)	1.902(0.007)	1.934(0.07)	1.928(0.007)
1/16	1.882(0.007)	1.841(0.007)	1.856(0.007)	1.897(0.007)	1.895(0.007)
1/32	1.821(0.007)	1.822(0.007)	1.831(0.007)	1.880(0.007)	1.880(0.007)
			K=140		
1	42.379(0.031)	39.793(0.026)	40.484(0.025)	40.021(0.026)	40.405(0.003)
1/2	41.097(0.029)	40.039(0.025)	40.394(0.025)	40.256(0.025)	40.357(0.025)
1/4	40.601(0.026)	40.188(0.025)	40.401(0.025)	40.373(0.025)	40.385(0.025)
1/8	40.449(0.025)	40.257(0.024)	40.396(0.024)	40.394(0.026)	40.392(0.025)
1/16	40.396(0.024)	40.276(0.024)	40.363(0.024)	40.363(0.024)	40.362(0.024)
1/32	40.377(0.024)	40.300(0.024)	40.350(0.024)	40.364(0.024)	40.363(0.024)

Table 3.2.: Option prices due to different simulation schemes, sample standard deviation given in parentheses

3.6. Outlook

Given (3.9) and substituting

$$\int_t^{t+\Delta} \sqrt{v_j(u)} e_j^\top dW(u) = \varepsilon_j^{-1} \left(v_j(t+\Delta) - v_j(t) - \kappa_j \theta_j \Delta + \int_t^{t+\Delta} v_j(u)\, du \right)$$

for $j = 1, ..., n$, the exact representation of the processes X_j is given by

$$\begin{aligned} & \ln\left(X_j(t+\Delta) + \alpha_j\right) \\ = \; & \ln\left(X_j(t) + \alpha_j\right) + -\frac{1}{2} \int_t^{t+\Delta} v_j \left|\beta_j\right|^2 dt + \rho_j \left|\beta_j\right| \varepsilon_j^{-1} \left(v_j(t+\Delta) - v_j(t) - \kappa_j \theta_j \Delta\right) \\ & + \rho_j \left|\beta_j\right| \varepsilon_j^{-1} \left(\int_t^{t+\Delta} v_j(u)\, du \right) + \int_t^{t+\Delta} \sqrt{1-\rho_j^2} \sqrt{v_j} \left|\beta_j\right| e_j^\top d\bar{W}. \end{aligned}$$

This substitution does indeed only work if we do not manipulate the unit vector e_j or manipulate it in both, the vola and the asset process. If we follow the second point and replace e_j by e_j^α the new correlations would read

$$\begin{aligned}
Cor_{X_j,X_{j'}} &= \frac{\rho_j\sqrt{v_j}\left|\beta_j\right|\left(e_j^\alpha\right)^T\rho_{j'}\sqrt{v_{j'}}\left|\beta_{j'}\right|e_{j'}^\alpha}{\sqrt{v_j\left|\beta_j\right|^2}\sqrt{v_{j'}\left|\beta_{j'}\right|^2}} \\
&\quad + \frac{\sqrt{1-\rho_j^2}\sqrt{v_j}\left|\beta_j\right|\left(e_j^\alpha\right)^T\sqrt{1-\rho_{j'}^2}\sqrt{v_{j'}}\left|\beta_{j'}\right|e_{j'}^\alpha}{\sqrt{v_j\left|\beta_j\right|^2}\sqrt{v_{j'}\left|\beta_{j'}\right|^2}} \\
&= \frac{\sqrt{v_jv_{j'}}\left|\beta_j\right|\left|\beta_{j'}\right|\left(\rho_j\rho_{j'}\left(e_j^\alpha\right)^T e_{j'}^\alpha+\sqrt{1-\rho_j^2}\sqrt{1-\rho_{j'}^2}\left(e_j^\alpha\right)^T e_{j'}^\alpha\right)}{\sqrt{v_j\left|\beta_j\right|^2}\sqrt{v_{j'}\left|\beta_{j'}\right|^2}} \\
&= \rho_j\rho_{j'}\left(e_j^\alpha\right)^T e_{j'}^\alpha+\sqrt{1-\rho_j^2}\sqrt{1-\rho_{j'}^2}\left(e_j^\alpha\right)^T e_{j'}^\alpha
\end{aligned}$$

$$Cor_{v_j,v_{j'}} := \left(e_j^\alpha\right)^T e_{j'}^\alpha,$$

$$\begin{aligned}
Cor_{X_j,v_{j'}} &= \frac{\rho_j\sqrt{v_j}\left|\beta_j\right|\left(e_j^\alpha\right)^T\varepsilon_{j'}\sqrt{v_{j'}}e_{j'}^\alpha}{\sqrt{v_j\left|\beta_j\right|^2}\sqrt{\varepsilon_{j'}^2v_{j'}}} \\
&= \rho_j\left(e_j^\alpha\right)^T e_{j'}^\alpha.
\end{aligned}$$

So one may use and manipulate e_j^α to restore $Cor_{v_j,v_{j'}}$.

This is important due to the fact that the approximation schemes proposed by Andersen for the one dimensional case are based on a moment matching idea for the first two moments. For the multidimensional case this would mean one has to take into account the covariances, too. Even though numerical experiments show, that the covariances are already fitted quite well we will nevertheless give a procedure for matching the covariances in addition. Let us therefor consider the TG scheme. The strategy to do so is as follows. The calculation of μ_j and σ_j is independent of the unit vector e_j, the only thing necessary from a theoretical point of view is that e_j is indeed a unit vector. Therefore choosing a different unit vector e_j^α would not change the value of μ_j and σ_j. After determining all μ_j and σ_j we will approximate the vola process by

$$\widehat{v}_j\left(t+\Delta\right) := (\mu_j+\sigma_j\left(e_j^\alpha\right)^T W)^+,$$

where e_j^α are unit vectors fitted to the covariance matrix with respect to the parameter α. The covariance is given by

$$\begin{aligned}
Cov\left(\widehat{v}_j\left(t+\Delta\right),\widehat{v}_k\left(t+\Delta\right)\left|\widehat{v}_j\left(t\right),\widehat{v}_k\left(t\right)\right.\right) &= \mathbb{E}\left[\widehat{v}_j\left(t+\Delta\right)\widehat{v}_k\left(t+\Delta\right)\left|\widehat{v}_j\left(t\right),\widehat{v}_k\left(t\right)\right.\right] \qquad (3.28) \\
&\quad - \mathbb{E}\left[\widehat{v}_j\left(t+\Delta\right)\left|\widehat{v}_j\left(t\right)\right.\right]\mathbb{E}\left[\widehat{v}_k\left(t+\Delta\right)\left|\widehat{v}_k\left(t\right)\right.\right],
\end{aligned}$$

where we have to calculate

$$\mathbb{E}\left[\widehat{v}_j\left(t+\Delta\right)\widehat{v}_k\left(t+\Delta\right)|\widehat{v}_j\left(t\right),\widehat{v}_k\left(t\right)\right]. \tag{3.29}$$

To the authors knowledge this is not known in general. This is a possible direction for future research to derive a formula for (3.29) for a parameter setting as general as possible. With (3.28) we have access to the correlation which we will match to a given correlation structure $\left(c_{jk}\right)_{jk}$. As an easy example we suggest to take $c_{jk} = e^{-\beta|j-k|}$. More involved examples taking into account more than one calibration variable could be taken into account (cf. Schoenmakers [2005]). This calibration taking place in each simulation step is a very time consuming part. Therefore we face the problem to derive computational tractable expressions of $Cov\left(\widehat{v}_j\left(t+\Delta\right),\widehat{v}_k\left(t+\Delta\right)|\widehat{v}_j\left(t\right),\widehat{v}_k\left(t\right)\right)$ (for the correlation, respectively) for the two schemes.

4. Multilevel simulation based policy iteration for optimal stopping – convergence and complexity

In this Chapter we present a novel approach to reduce the complexity of simulation based policy iteration methods for solving optimal stopping problems. When constructing an improved policy using Monte Carlo methods one typically ends up with a nested simulation procedure. We use here the multilevel idea in the context of the nested simulations, where each level corresponds to a specific number of inner simulations. A main concern is dedicated to a thorough analysis of the convergence rates in the multilevel policy improvement algorithm. In a detailed complexity analysis we show that a significant reduction in computational effort can be achieved in comparison to the standard Monte Carlo based policy iteration. We want to remark that all Proofs are deferred to Section 4.6 for reasons of readability.

4.1. Policy iteration for optimal stopping

We work under a similar setup as in Section 1.1, this time assuming we are given a constant interest rate given by r as discussed at the end of that section. We are using a slightly different notation that seems to be more convenient in case of the discrete setup used here.

In this section we review the (probabilistic) policy iteration (improvement) method for the optimal stopping problem in discrete time. For illustration, we formalize this in the context of pricing an American (Bermudan) derivative. We will work in a stylized setup where $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space with discrete filtration $\mathbb{F} = (\mathcal{F}_j)_{j=0,\ldots,T}$ for $T \in \mathbb{N}_+$. An American derivative on a nonnegative adapted cash-flow process $(Z_j)_{j=0,\ldots,T}$ entitles the holder to exercise or receive cash Z_j at an exercise time $j \in \{0, ..., T\}$ that may be chosen once. It is assumed that Z_j is expressed in units of some specific pricing numéraire N with $N_0 := 1$ (without loss of generality we may take $N \equiv 1$). Then the value of the American option at time $j \in \{0, ..., T\}$ (in units of the numéraire) is given by the solution of the optimal stopping problem:

$$Y_j^* := \underset{\tau \in \mathcal{T}[j,\ldots,T]}{\text{ess.sup}} \, \mathbb{E}^{\mathcal{F}_j} \left[Z_\tau \right], \tag{4.1}$$

provided that the option is not exercised before j. In (4.1) $\mathcal{T}[j, \ldots, T]$ is the set of $\mathbb{F}$-stopping times taking values in $\{j, \ldots, T\}$ and the process $\left(Y_j^*\right)_{j=0,\ldots,T}$ is called the Snell envelope. A stopping time τ_j^* satisfying

$$Y_j^* = \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_j^*}\right]$$

is called an optimal stopping time and $\left(\tau_j^*\right)_{j=0,\ldots,T}$ an optimal stopping family. The Snell envelope Y^* is a supermartingale by

$$\mathbb{E}^{\mathcal{F}_j}\left[Y_k^*\right] = \mathbb{E}^{\mathcal{F}_j}\left[\mathbb{E}^{\mathcal{F}_k}\left[Z_{\tau_k^*}\right]\right] = \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_k^*}\right] \le \sup_{\tau\in\{j,\ldots,T\}} \mathbb{E}^{\mathcal{F}_j}\left[Z_\tau\right] = Y_j^*, \quad \text{for } j < k.$$

It also satisfies the backward dynamic programming equation (Bellman principle)

$$Y_j^* = \max\left(Z_j, \mathbb{E}^{\mathcal{F}_j}[\left[Y_{j+1}^*\right]\right), \quad 0 \le j < T, \quad Y_T^* = Z_T.$$

An optimal stopping family is than given by

$$\tau_j^* = \inf\left\{k : j \le k < T, Z_k > Y_k^*\right\} \wedge T, \quad j = 0, ..., T.$$

An exercise policy is a family of stopping times $(\tau_j)_{j=0,\ldots,T}$ such that $\tau_j \in \mathcal{T}[j,\ldots,T]$. To further formalize we state the following definitions.

Definition 41 *An exercise policy $(\tau_j)_{j=0,\ldots,T}$ is said to be **consistent** if*

$$\tau_j > j \Longrightarrow \tau_j = \tau_{j+1}, \quad 0 \le j < T, \quad \textit{and} \quad \tau_T = T. \tag{4.2}$$

Given a consistent stopping family natural questions are whether this stopping family is optimal and if not how to improve it. These questions can be partially answered by policy iteration. Starting with some input stopping time often given by trivial examples like $\tau_j \equiv j$ or $\tau_j \equiv T$ we define the process $(Y_j)_{j=0,\ldots,T}$ by

$$Y_j := \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_j}\right]. \tag{4.3}$$

Due to the usually suboptimal input stopping family $(Y_j)_{j=0,\ldots,T}$ is a lower bound approximation of $\left(Y_j^*\right)_{j=0,\ldots,T}$. For the construction of an improved stopping family let us consider the process $\left(\tilde{Y}_j\right)_{j=0,\ldots,T}$ given by

$$\tilde{Y}_j := \max_{p:j<p\le\min(j+\kappa,T)} \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_p}\right]$$

for some integer valued window parameter κ, $1 \le \kappa \le T$. We will here consider $\kappa = 1$, so $\min(j+\kappa, T) = j + 1 \wedge T$. Using $\tilde{Y}_j$ as an exercise criterion a new stopping family is given by

$$\hat{\tau}_j = \inf\left\{k : j \le k < T, Z_k > \mathbb{E}^{\mathcal{F}_k}\left[Z_{\tau_{k+1}}\right]\right\} \wedge T, \quad j = 0, ..., T.$$

Let us further consider the process

$$\widehat{Y}_j := \mathbb{E}^{\mathcal{F}_j}\left[Z_{\widehat{\tau}_j}\right] \tag{4.4}$$

as a next approximation to the Snell envelope, compare Theorem 43.

Definition 42 ***(standard) policy iteration***
Given a consistent stopping family $(\tau_j)_{j=0,...,T}$ let us consider a new family $(\widehat{\tau}_j)_{j=0,...,T}$ defined by

$$\widehat{\tau}_j = \inf\left\{k : j \leq k < T, Z_k > \mathbb{E}^{\mathcal{F}_k}\left[Z_{\tau_{k+1}}\right]\right\} \wedge T, \quad j = 0, ..., T \tag{4.5}$$

with $\wedge$ denoting the minimum operator and $\inf \varnothing := +\infty$. *The new family $(\widehat{\tau}_j)_{j=0,...,T}$ is termed a policy iteration of $(\tau_j)_{j=0,...,T}$.*

$(\widehat{\tau}_j)_{j=0,...,T}$ is again a consistent stopping family. The following Theorem states that $\left(\widehat{Y}_j\right)_{j=0,...,T}$ usually is an improvement of $(Y_j)_{j=0,...,T}$.

Theorem 43 *Let $(\tau_j)_{j=0,...,T}$ be a consistent stopping family satisfying (4.2) where we define $(Y_j)_{j=0,...,T}$ and $\left(\widehat{Y}_j\right)_{j=0,...,T}$ by (4.3), and (4.4). It holds*

$$Y_j \leq \widehat{Y}_j \leq Y_j^*, \quad j = 0, ..., T.$$

Proof. The second inequality is trivially fulfilled due to the optimality of Y_j^*. For the first inequality it holds

$$Y_T = \widehat{Y}_T = Z_T.$$

We proof by backward induction. For that suppose $Y_j \leq \widehat{Y}_j$ has already be shown. It then follows

$$\begin{aligned}
\widehat{Y}_{j-1} = \mathbb{E}^{\mathcal{F}_{j-1}}\left[Z_{\widehat{\tau}_{j-1}}\right] &= I_{\left\{\widehat{\tau}_{j-1}=j-1\right\}} Z_{j-1} + I_{\left\{\widehat{\tau}_{j-1}>j-1\right\}} \mathbb{E}^{\mathcal{F}_{j-1}}\left[\mathbb{E}^{\mathcal{F}_j}\left[Z_{\widehat{\tau}_j}\right]\right] \\
&= I_{\left\{\widehat{\tau}_{j-1}=j-1\right\}} Z_{j-1} + I_{\left\{\widehat{\tau}_{j-1}>j-1\right\}} \mathbb{E}^{\mathcal{F}_{j-1}}\left[\widehat{Y}_j\right] \\
&= Z_{j-1} + I_{\left\{\widehat{\tau}_{j-1}>j-1\right\}}\left(\mathbb{E}^{\mathcal{F}_{j-1}}\left[\widehat{Y}_j\right] - Z_{j-1}\right) \\
&\geq Y_{j-1} + I_{\left\{\widehat{\tau}_{j-1}>j-1\right\}}\left(\mathbb{E}^{\mathcal{F}_{j-1}}\left[\widehat{Y}_j\right] - Z_{j-1}\right), \quad \widehat{\tau}_{j-1} = j-1 \\
&\geq Y_{j-1} + I_{\left\{\widehat{\tau}_{j-1}>j-1\right\}}\left(\mathbb{E}^{\mathcal{F}_{j-1}}\left[Y_j\right] - Z_{j-1}\right), \quad \widehat{Y}_j \geq Y_j \\
&\geq Y_{j-1}, \quad \mathbb{E}^{\mathcal{F}_{j-1}}\left[Y_j\right] - Z_{j-1} > 0 \text{ on } \left\{\widehat{\tau}_{j-1} > j-1\right\}.
\end{aligned}$$

■

Naturally we want to extend the one-step improvement as stated in Definition 42 iteratively. Starting with a consistent stopping family $\left(\tau_j^{(0)}\right)_{j=0,...,T}$ we define pairs

$$\left(\left(\tau_j^{(m)}\right)_{j=0,...,T}, \left(Y_j^{(m)}\right)_{j=0,...,T}\right)_{m=0,1,...}$$

in the following way, where

$$Y_j^{(m)} := \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_j^{(m)}}\right]. \tag{4.6}$$

Suppose that we already constructed $\left(\left(\tau_j^{(m)}\right)_{j=0,...,T}, \left(Y_j^{(m)}\right)_{j=0,...,T}\right)$ for $m \geq 0$. Then we define

$$\tau_j^{(m+1)} = \inf\left\{k : j \leq k < T, Z_k > \mathbb{E}^{\mathcal{F}_k}\left[Z_{\tau_{k+1}^{(m)}}\right]\right\} \wedge T, \quad j = 0, ..., T.$$

Due to Theorem 43 the following holds.

$$Y_j^{(0)} \leq Y_j^{(m)} \leq Y_j^{(m+1)} \leq Y_j^*, \quad m \geq 0, \ j = 0, ..., T \tag{4.7}$$

Lemma 44 *Denote*

$$\tilde{Y}_j = \max_{p: j < p \leq \min(j+\kappa, T)} \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_p}\right]$$

with respect to a policy iterated stopping family $(\widehat{\tau}_j)_{j=0,...,T}$. It holds

$$\widehat{Y}_j \geq Z_j, \quad j = 0, ..., T.$$

Proof. Suppose the opposite. We then have

$$\begin{aligned}
&Z_j > \widehat{Y}_j \Rightarrow Z_j > Y_j, \\
&Z_j > Z_j \text{ on } \{\tau_j = j\} \Rightarrow \lightning, \\
&Z_j > \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}}\right] \text{ on } \{\tau_j > j\} \Rightarrow \widehat{\tau}_j = j \Rightarrow Z_j = \widehat{Y}_j \Rightarrow \lightning.
\end{aligned}$$

■

To show the optimality of the multi-step improvement we need the following Lemma.

Lemma 45 *For $m \geq 0$ and $j = 0, ..., T$ we have*

$$Y_j^{(m+1)} \geq \mathbb{E}^{\mathcal{F}_j}\left[Y_{j+1}^{(m)}\right].$$

Proof. Let us first rewrite the right hand side of the inequality using (4.7) and tower property.

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m+1)}}\right] &= \mathbb{E}^{\mathcal{F}_j}\left[\mathbb{E}^{\mathcal{F}_{j+1}}\left[Z_{\tau_{j+1}^{(m+1)}}\right]\right] \geq \mathbb{E}^{\mathcal{F}_j}\left[\mathbb{E}^{\mathcal{F}_{j+1}}\left[Z_{\tau_{j+1}^{(m)}}\right]\right] \\
&= \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m)}}\right]
\end{aligned}$$

We further have

$$
\begin{aligned}
Y_j^{(m+1)} &= I_{\left\{\tau_j^{(m+1)}=j\right\}} Z_j + I_{\left\{\tau_j^{(m+1)}>j\right\}} \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m+1)}}\right] \\
&\geq I_{\left\{\tau_j^{(m+1)}=j\right\}} Z_j + I_{\left\{\tau_j^{(m+1)}>j\right\}} \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m)}}\right] \\
&\pm I_{\left\{\tau_j^{(m+1)}=j\right\}} \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m)}}\right] \\
&= I_{\left\{\tau_j^{(m+1)}=j\right\}} \left(Z_j - \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m)}}\right]\right) + \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m)}}\right] \\
&\geq \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}^{(m)}}\right], \quad Z_j - \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_j^{(m)}}\right] > 0 \text{ on } \left\{\tau_j^{(m+1)} = j\right\} \\
&= \mathbb{E}^{\mathcal{F}_j}\left[Y_{j+1}^{(m)}\right].
\end{aligned}
$$

■

Finally we want to show that the policy iteration achieves the Snell envelope after a finite number of iterations.

Proposition 46 *For $m \geq 0$ and $j = 0, ..., T$ we have*

$$Y_j^{(m)} = Y_j^*, \quad m \geq T - j.$$

Proof. It is clear that

$$Y_T^{(m)} = Y_T^*.$$

Suppose we have

$$Y_j^{(m)} = Y_j^*$$

for some j. Now for $m \geq T - (j-1)$, so $m \geq 1$, we have using Lemma 45 in the first and Lemma 44 in the second inequality we have

$$
\begin{aligned}
&Y_{j-1}^{(m)} \geq \mathbb{E}^{\mathcal{F}_{j-1}}\left[Y_j^{(m-1)}\right] = \mathbb{E}^{\mathcal{F}_{j-1}}\left[Y_j^*\right] \\
&Y_{j-1}^{(m)} \geq Z_{j-1}, \quad m \geq 1 \\
&Y_{j-1}^{(m)} \geq I_{\{\tau_j^*=j-1\}} Z_{j-1} + I_{\{\tau_j^*>j-1\}} \mathbb{E}^{\mathcal{F}_{j-1}}\left[Y_j^*\right] = Y_{j-1}^*.
\end{aligned}
$$

■

See also Bender and Schoenmakers [2006] for a further analysis regarding stability issues, and extensions to policy iteration methods for multiple stopping.

4.2. Simulation based policy iteration

In order to apply the policy iteration method in practice, we henceforth assume that the cash-flow Z_j is of the form (while slightly abusing of notation) $Z_j = Z_j(X_j)$ for some underlying (possibly high-dimensional) Markovian process X. As a consequence, the Snell envelope process then has the Markovian form $Y_j^* = Y_j^*(X_j), \quad j = 0, ..., T$, as well. Furthermore, it is assumed that a consistent stopping family (τ_j) depends on ω only through the path $X_\cdot$ in the following way: For each j the event $\{\tau_j = j\}$ is measurable with respect to X_j, and τ_j is measurable with respect to $(X_k)_{j \le k \le T}$, i.e.

$$\tau_j(\omega) = h_j(X_j(\omega), ..., X_T(\omega)) \tag{4.8}$$

for some Borel measurable function h_j. A typical example of such a stopping family is

$$\tau_j = \inf\{k : j \le k \le T, \quad Z_k(X_k) \ge f_k(X_k)\}$$

for a set of real valued functions $f_k(x)$. The next issue is the estimation of the conditional expectations in (4.5). A canonical approach is the use of sub simulations. In this respect we consider an enlarged probability space $(\Omega, \mathcal{F}', \mathbb{F}', \mathbb{P})$, where $\mathbb{F}' = (\mathcal{F}'_j)_{j=0,...,T}$ and $\mathcal{F}_j \subset \mathcal{F}'_j$ for each j. By assumption, $\mathcal{F}'_j$ specified as

$$\mathcal{F}'_j = \mathcal{F}_j \vee \sigma\left\{X_\cdot^{i,X_i}, \ i \le j,\right\} \text{ with } \mathcal{F}_j = \sigma\left\{X_i, \ i \le j\right\},$$

where for a generic $(\omega, \omega_{in}) \in \Omega$, $X_\cdot^{i,X_i} := X_k^{i,X_i(\omega)}(\omega_{in})$, $k \ge i$ denotes a sub trajectory starting at time i in the state $X_i(\omega) = X_i^{i,X_i(\omega)}$ of the outer trajectory $X(\omega)$. In particular, the random variables $X_\cdot^{i,X_i}$ and $X_\cdot^{i',X_{i'}}$ are by assumption independent, conditionally $\{X_i, X_{i'}\}$, for $i \neq i'$. On the enlarged space we consider $\mathcal{F}'_j$ measurable estimations $\mathcal{C}_{j,M}$ of $C_j := \mathbb{E}^{\mathcal{F}_j}\left[Z_{\tau_{j+1}}\right]$ as being standard Monte Carlo estimates based on M sub simulations. More precisely, for

$$C_j(X_j) := \mathbb{E}_{X_j}\left[Z_{\tau_{j+1}}\right]$$

define

$$\mathcal{C}_{j,M} := \frac{1}{M}\sum_{m=1}^{M} Z_{\tau_{j+1}^{(m)}}(X_{\tau_{j+1}^{(m)}}^{j,X_j,(m)}),$$

where the stopping times

$$\tau_{j+1}^{(m)} := h_{j+1}(X_{j+1}^{j,X_j,(m)}, ..., X_T^{j,X_j,(m)})$$

(cf. (4.8)) are evaluated on sub-trajectories $X_\cdot^{j,X_j,(m)}$, $m = 1, ..., M$, all starting at time j in X_j. Obviously, $\mathcal{C}_{j,M}$ is an unbiased estimator for C_j with respect to $\mathbb{E}^{\mathcal{F}_j}[\cdot]$. We thus end up with a simulation based version of (4.5),

$$\widehat{\tau}_{j,M} = \min\left\{k : j \le k < T, \ Z_k > \mathcal{C}_{k,M}\right\} \wedge T.$$

Now set

$$\widehat{Y}_{j,M} := \mathbb{E}^{\mathcal{F}_j}[Z_{\widehat{\tau}_{j,M}}].$$

Next we analyze the bias and the variance of the estimator $\widehat{Y}_{0,M}$.

Proposition 47 *Suppose that* $|Z_j| < B$ *for some* $B > 0$. *Let us further assume that there exist a constant* $D > 0$ *and* $\alpha > 0$, *such that for any* $\delta > 0$ *and* $j = 0, \ldots, T-1$,

$$\mathbb{P}(|C_j - Z_j| \leq \delta) \leq D\delta^{\alpha}. \tag{4.9}$$

It then holds,

$$\mathbb{P}\left(\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\right) \leq D_1 M^{-\alpha/2} \tag{4.10}$$

for some constant $D_1 > 0$.

Corollary 48 *Under the assumptions of Proposition 47, it follows immediately by (4.10) that*

$$\widehat{Y}_{0,M} - \widehat{Y}_0 = O(M^{-\alpha/2}) \quad \textit{and} \quad \mathbb{E}[\Big(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0}\Big)^2] = O(M^{-\alpha/2}).$$

Proof. We have

$$\widehat{Y}_{0,M} - \widehat{Y}_0 = \mathbb{E}[\Big(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0}\Big)\mathbf{1}_{\{\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\}}] \leq B\mathbb{P}\left(\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\right)$$

and

$$\mathbb{E}[\Big(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0}\Big)^2] = \mathbb{E}[\Big(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0}\Big)^2\mathbf{1}_{\{\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\}}] \leq B^2\mathbb{P}\left(\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\right).$$

■

Under somewhat more restrictive assumptions than the ones of Proposition 47 we can prove the following theorem.

Theorem 49 *Suppose that*

(i) *the transition kernels of the chain* (X_i) *are infinitely differentiable with bounded derivatives of any order;*

(ii) *the cash-flow is bounded, i.e. there exists a constant* B *such that* $|Z_j(x)| < B$ *a.s. for all* x;

(iii) *the function*

$$\sigma_j^2(x) := \mathbb{E}\left[\Big(Z_{\tau_{j+1}}(X_{\tau_{j+1}}^{j,x}) - C_j(x)\Big)^2\right] = Var\left[Z_{\tau_{j+1}}(X_{\tau_{j+1}}^{j,x})\right]$$

is bounded (due to (i)) and bounded away from zero uniformly in x *and* j;

(iv) *the density of the random variable*

$$Z_j(X_j) - C_j(X_j)$$

conditional on $\mathcal{F}_{j-1}$, *i.e. given* $X_{j-1} = x_{j-1}$, *is of the form* $x \to h(x; x_{j-1})$, *where* $h(\cdot; x_{j-1})$ *is at least two times differentiable for each* x_{j-1}.

Then it holds

$$\left|\widehat{Y}_{0,M} - \widehat{Y}_0\right| = O(M^{-1}), \quad M \to \infty.$$

Discussion Theorem 49 controls the bias of the estimator $\widehat{Y}_{0,M}$ for the lower approximation $\widehat{Y}_0$ to the Snell envelope due to the improved policy $(\widehat{\tau}_j)$. Concerning the difference between $\widehat{Y}_0$ and Y_0, we infer from Kolodko and Schoenmakers [2006], Lemma 4.5, that

$$0 \leq \widehat{Y}_0 - Y_0 \leq \mathbb{E} \sum_{k=\tau_0}^{\widehat{\tau}_0 - 1} [\mathbb{E}^{\mathcal{F}_k} Y_{k+1} - Y_k]$$

(where automatically $\widehat{\tau}_0 \geq \tau_0$ when (τ_j) is consistent). Hence, for a bounded cash-flow process with $|Z_j| < B$ we get

$$0 \leq \widehat{Y}_0 - Y_0 \leq TB\mathbb{P}(\tau_j \neq \widehat{\tau}_j) \leq TB\mathbb{P}(\tau_j \neq \tau_j^*),$$

as $\tau_j = \tau_j^*$ implies $\tau_j = \widehat{\tau}_j = \tau_j^*$. If $\mathbb{P}(\tau_j \neq \tau_j^*) = 0$, we get $Y_0 = \widehat{Y}_0 = Y_0^*$.

4.3. Standard Monte Carlo approach

Within Markovian setup as introduced in Section 4.2, consider for some fixed natural numbers N and M, the estimator:

$$\widehat{\mathcal{Y}}_{N,M} := \frac{1}{N} \sum_{n=1}^{N} Z_{\widehat{\tau}_M}^{(n)} \tag{4.11}$$

for $\widehat{Y}_M := \widehat{Y}_{0,M}$ with $\widehat{\tau}_M := \widehat{\tau}_{0,M}$, based on n realizations $Z_{\widehat{\tau}_M}^{(n)}$, $n = 1, ..., N$, of the stopped cash-flow $Z_{\widehat{\tau}_M}$. Let us investigate the complexity, i.e. the required computational costs, in order to compute $\widehat{Y} := \widehat{Y}_0$ with a prescribed (root-mean-square) accuracy ϵ, by using the estimator (4.11). Under the assumptions of Corollary 48 we have with $\gamma = \alpha/2$, or $\gamma = 1$ if Theorem 49 applies, for the mean squared error,

$$\begin{aligned} \mathbb{E}\left[\widehat{\mathcal{Y}}_{N,M} - \widehat{Y}\right]^2 &\leq N^{-1}\mathrm{Var}\left[Z_{\widehat{\tau}_M}\right] + \left|\widehat{Y} - \widehat{Y}_M\right|^2 \\ &\leq N^{-1}\sigma_\infty^2 + \mu_\infty^2 M^{-2\gamma}, \quad M \geq M_0, \end{aligned} \tag{4.12}$$

for some constants μ_∞ and $\sigma_\infty^2 := \sup_{M \geq M_0} \mathrm{Var}\left[Z_{\widehat{\tau}_M}\right]$, where M_0 denotes some fixed minimum number of sub trajectories used for computing the stopping time $\widehat{\tau}_M$. In order to bound (4.12) by ϵ^2, we set

$$M = \left\lceil \left(\frac{2^{1/2}\mu_\infty}{\epsilon} \right)^{1/\gamma} \right\rceil, \quad N = \left\lceil \frac{2\sigma_\infty^2}{\epsilon^2} \right\rceil$$

with $\lceil x \rceil$ denoting the smallest integer bigger or equal than x. For notational simplicity we will henceforth omit the brackets and carry out calculations with generally non-integer M, N. This will neither affect complexity rates nor the asymptotic proportionality constants. Thus the computational complexity for reaching accuracy ϵ when $\epsilon \downarrow 0$ is given by

$$\mathcal{C}^{N,M}_{\text{stand}}(\epsilon) := NM = \frac{2\sigma^2_\infty \left(2^{1/2}\mu_\infty\right)^{1/\gamma}}{\epsilon^{2+1/\gamma}}, \tag{4.13}$$

where, again for simplicity, it is assumed that both the cost of simulating one outer trajectory and one sub trajectory is equal to one unit. In typical applications we have $\gamma = 1$ and the complexity of the standard Monte Carlo method is of order $O(\epsilon^{-3})$. However, if $\gamma = 1/2$ the complexity is as high as $O(\epsilon^{-4})$.

4.4. Multilevel Monte Carlo approach

For a fixed natural number L and a sequence of natural numbers $\mathbf{m} := (m_0, ..., m_L)$ satisfying $1 \leq m_0 < ... < m_L$, we consider in the spirit of Giles [2008] the telescoping sum:

$$\widehat{Y}_{m_L} = \widehat{Y}_{m_0} + \sum_{l=1}^{L} \left(\widehat{Y}_{m_l} - \widehat{Y}_{m_{l-1}}\right). \tag{4.14}$$

Further we approximate the expectations $\widehat{Y}_{m_l}$ in (4.14). We take a set of natural numbers $\mathbf{n} := (n_0, ..., n_L)$ satisfying $n_0 > ... > n_L \geq 1$, and simulate the initial set of cash-flows

$$\left\{Z^{(j)}_{\widehat{\tau}_{m_0}}, \quad j = 1, ..., n_0\right\},$$

due to the initial set of trajectories $X^{0,x,(j)}_{\cdot}$, $j = 1, ..., n_0$. Next we simulate *independently* for each level $l = 1, ..., L$, a set of pairs

$$\left\{(Z^{(j)}_{\widehat{\tau}_{m_l}}, Z^{(j)}_{\widehat{\tau}_{m_{l-1}}}), \quad j = 1, ..., n_l\right\}$$

due to a set of trajectories $X^{0,x,(j)}_{\cdot}$, $j = 1, ..., n_l$, to obtain a multilevel estimator

$$\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0}\sum_{j=1}^{n_0} Z^{(j)}_{\widehat{\tau}_{m_0}} + \sum_{l=1}^{L} \frac{1}{n_l}\sum_{j=1}^{n_l}\left(Z^{(j)}_{\widehat{\tau}_{m_l}} - Z^{(j)}_{\widehat{\tau}_{m_{l-1}}}\right) \tag{4.15}$$

as an approximation to $\widehat{Y}$ (cf. Belomestny and Schoenmakers [2011]). Henceforth we always take $\mathbf{m}$ to be a geometric sequence, i.e., $m_l = m_0\kappa^l$, for some m_0, $\kappa \in \mathbb{N}$, $\kappa \geq 2$.

Complexity analysis

Let us now study the complexity of the multilevel estimator (4.15) under the assumption that the conditions of Proposition 47 or Theorem 49 are fulfilled. For the bias we have

$$\left|\mathbb{E}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}}\right] - \widehat{Y}\right| = \left|\mathbb{E}\left[Z_{\widehat{\tau}_{m_L}} - Z_{\widehat{\tau}}\right]\right| \leq \mu_\infty m_L^{-\gamma}, \tag{4.16}$$

and for the variance it holds

$$\operatorname{Var}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}}\right] = \frac{1}{n_0}\operatorname{Var}\left[Z_{\widehat{\tau}_{m_0}}\right] + \sum_{l=1}^{L}\frac{1}{n_l}\operatorname{Var}\left[Z_{\widehat{\tau}_{m_l}} - Z_{\widehat{\tau}_{m_{l-1}}}\right],$$

where due to Proposition 47, the terms with $l > 0$ may be estimated by

$$\begin{aligned}\operatorname{Var}\left[Z_{\widehat{\tau}_{m_l}} - Z_{\widehat{\tau}_{m_{l-1}}}\right] &\leq \mathbb{E}\left[\left(Z_{\widehat{\tau}_{m_l}} - Z_{\widehat{\tau}_{m_{l-1}}}\right)^2\right] \\ &\leq 2\mathbb{E}\left[\left(Z_{\widehat{\tau}_{m_l}} - Z_{\widehat{\tau}}\right)^2\right] + 2\mathbb{E}\left[\left(Z_{\widehat{\tau}_{m_{l-1}}} - Z_{\widehat{\tau}}\right)^2\right] \\ &\leq C\left(m_l^{-\beta} + m_{l-1}^{-\beta}\right) \leq C m_l^{-\beta}\left(1+\kappa^{\beta}\right) \leq \mathcal{V}_\infty m_l^{-\beta}, \end{aligned} \tag{4.17}$$

with $\beta := \alpha/2$, and suitable constants C, $\mathcal{V}_\infty$. In typical applications, we have that $C_j - Z_j$ in (4.9) has a positive but non-exploding density in zero which implies $\alpha = 1$, hence $\beta = 1/2$. This rate is confirmed by numerical experiments. Henceforth, we assume $\beta < 1$.

We are now going to analyze the optimal complexity of the multilevel algorithm. Our optimization approach is based on a separate treatment of n_0 and $n_i,\ i = 1, ..., L$. In particular, we assume that

$$n_l = n_1 \kappa^{(1+\beta)/2 - l(1+\beta)/2}, \quad 1 \leq l \leq L,$$

where the integers n_0 and n_1 are to be determined, and for the sub-simulations we take

$$m_l = m_0 \kappa^l, \quad 0 \leq l \leq L.$$

We further reuse the sub-simulations related to m_{l-1} for the computation of $\widehat{Y}_{m_l}$ so that the multilevel complexity becomes

$$\begin{aligned}\mathcal{C}_{ML}^{\mathbf{n},\mathbf{m}} &= n_0 m_0 + \sum_{l=1}^{L} n_l m_l \\ &= n_0 m_0 + n_1 m_0 \kappa \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1}. \end{aligned} \tag{4.18}$$

Theorem 50 *The asymptotic complexity of the multilevel estimator $\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}}$ for $0 < \beta < 1$ is given by*

$$\mathcal{C}_{ML}^* := \mathcal{C}_{ML}\left(n_0^*, n_1^*, L^*, m_0, \epsilon\right) :=$$
$$\frac{(1-\beta)\,\mathcal{V}_\infty \mu_\infty^{(1-\beta)/\gamma}}{2\gamma\left(1-\kappa^{-(1-\beta)/2}\right)^2}\left(1 + 2\gamma/\left(1-\beta\right)\right)^{1+(1-\beta)/(2\gamma)}\left(1 + O\left(\epsilon^{(1-\beta)/(2\gamma)}\right)\right)\epsilon^{-2-(1-\beta)/\gamma}, \tag{4.19}$$

where the optimal values n_0^*, n_1^*, L^* *have to be chosen according to*

$$n_0^* := n_0^*(L^*, m_0, \epsilon) :=$$
$$\frac{\sigma_\infty \mathcal{V}_\infty^{1/2} \mu_\infty^{(1-\beta)/(2\gamma)} (1-\beta)}{2\gamma m_0^{1/2} \left(1-\kappa^{-(1-\beta)/2}\right)} (1 + 2\gamma/(1-\beta))^{1+(1-\beta)/(4\gamma)} \times$$
$$\epsilon^{-2-(1-\beta)/(2\gamma)} \left(1 + O\left(\epsilon^{(1-\beta)/(2\gamma)}\right)\right) \quad \textit{and,} \tag{4.20}$$
$$n_1^* := n_1^*(L^*, m_0, \epsilon) :=$$
$$\frac{\mathcal{V}_\infty \mu_\infty^{(1-\beta)/(2\gamma)} (1-\beta)}{2\gamma m_0^{(1+\beta)/2} \left(1-\kappa^{-(1-\beta)/2}\right)} (1 + 2\gamma/(1-\beta))^{1+(1-\beta)/(4\gamma)} \kappa^{-(1+\beta)/2} \times$$
$$\epsilon^{-2-(1-\beta)/(2\gamma)} \left(1 + O\left(\epsilon^{(1-\beta)/(2\gamma)}\right)\right) \quad \textit{and,} \tag{4.21}$$
$$L^* := \frac{\ln \epsilon^{-1} + \ln\left[\frac{\mu_\infty}{m_0^\gamma} (1 + 2\gamma/(1-\beta))^{1/2}\right]}{\gamma \ln \kappa} + O\left(\epsilon^{(1-\beta)/(2\gamma)}\right). \tag{4.22}$$

Note that, asymptotically, the optimal complexity $\mathcal{C}_{ML}^*$ is independent of m_0. We therefore propose to choose m_0 by experience. In typical numerical examples $m_0 = 100$ turns out to be a robust choice.

Discussion For the standard algorithm given optimally chosen M^*, N^* we have the complexity given by (4.13), so the gain ratio of the multilevel approach over the standard Monte Carlo algorithm is asymptotically given by

$$\mathcal{R}^*(\epsilon) := \frac{\mathcal{C}_{ML}^*(\epsilon)}{\mathcal{C}_{\text{stan}}^{N^*,M^*}(\epsilon)}$$
$$\sim \frac{(1-\beta)(1 + 2\gamma/(1-\beta))^{1+(1-\beta)/(2\gamma)} \mathcal{V}_\infty}{2^{2+1/(2\gamma)} \gamma \left(1-\kappa^{-(1-\beta)/2}\right)^2 \sigma_\infty^2 \mu_\infty^{\beta/\gamma}} \epsilon^{\beta/\gamma}, \quad \epsilon \downarrow 0. \tag{4.23}$$

For the variance and bias rate β and γ, respectively, cf. (4.17) and (4.16). Typically, we have that $\beta = 1/2$ and that $\gamma \geq 1/2$, where the value of γ depends on whether Theorem 49 applies or not. In any case we may conclude that the smaller γ the larger the complexity gain.

4.5. Numerical comparison of the two estimators

In this section we will compare both algorithms in a numerical example. The usual way would be to take both algorithms, take optimal parameters and compare the complexities given an accuracy ϵ, like we did in the previous section in general. The optimal parameters depend on knowledge of some quantities, e.g. the coefficients of the bias rates. This knowledge might be gained by pre-computation (based on relatively smaller sample sizes) for instance. Here we propose a more pragmatic and robust approach (cf. Belomestny and Schoenmakers [2011]).

Let us assume that a practitioner knows his standard algorithm well and provides us with his "optimal"M (inner simulations), N (outer simulations). So his computational budget amounts to MN. Given the same budget MN we are now going to configure the multilevel estimator such that $m_L = M$, i.e. the bias is the same for both algorithms. We next show that n_0, n_1, and L can be chosen in such a way that the variance of the multilevel estimator is significantly below the variance of the standard one. Although this approach will not achieve the optimal gain (4.23) for $\epsilon \downarrow 0$ (hence for $M \to \infty$), it has the advantage that we may compare the accuracy of the multilevel estimator with the standard one for any fixed M and arbitrary N. The details are spelled out below.

Taking

$$M = m_L = m_0 \kappa^L \tag{4.24}$$

we have for the biases

$$\mathbb{E}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} - \widehat{Y}\right] = \mathbb{E}\left[\widehat{\mathcal{Y}}_{N,M} - \widehat{Y}\right] \leq \frac{\mu_\infty}{M^\gamma}.$$

As stated above we assume the same computational budget for both algorithms leading to the following constraint (see (4.18))

$$NM = n_0 m_0 + n_1 m_0 \kappa \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1}.$$

Let us write for $\xi \in \mathbb{R}_+$,

$$\begin{aligned} n_1 &:= \xi n_0, \\ n_0 &= \frac{NM}{m_0 + \xi m_0 \kappa \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1}}. \end{aligned} \tag{4.25}$$

With (4.24) and (4.25) we have for the variance estimate (4.36)

$$\begin{aligned} \operatorname{Var}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}}\right] &\leq \frac{\sigma_\infty^2}{n_0} + \frac{\mathcal{V}_\infty \kappa^{-\beta}}{\xi n_0 M^\beta \kappa^{-\beta L}} \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} \\ &= \frac{\sigma_\infty^2 \kappa^{-L}}{N} \left(1 + \frac{\mathcal{V}_\infty \kappa^{-\beta+\beta L}}{\xi M^\beta \sigma_\infty^2} \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1}\right) \\ &\times \left(1 + \xi \kappa \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1}\right) \\ &= \frac{\sigma_\infty^2 \kappa^{-L}}{N} \left(1 + \frac{a}{\xi}\right)(1 + b\xi) \end{aligned} \tag{4.26}$$

Expression (4.26) attains its minimum at

$$\xi^\circ := \sqrt{\frac{a}{b}} = \frac{\mathcal{V}_\infty^{1/2} \kappa^{(-\beta-1+\beta L)/2}}{M^{\beta/2} \sigma_\infty}, \tag{4.27}$$

which gives the "optimal" values n_0° and n_1° via (4.25), and

$$\begin{aligned}\operatorname{Var}\left[\widehat{\mathcal{Y}}_{\mathbf{n}^\circ,\mathbf{m}}\right] &\leq \frac{\sigma_\infty^2\kappa^{-L}}{N}\left(1+\sqrt{ab}\right)^2 \\ &= \frac{\sigma_\infty^2\kappa^{-L}}{N}\left(1+\frac{\kappa^{L(1-\beta)/2}-1}{\kappa^{(1-\beta)/2}-1}\frac{\mathcal{V}_\infty^{1/2}\kappa^{(1-\beta+\beta L)/2}}{M^{\beta/2}\sigma_\infty}\right)^2.\end{aligned}$$

The ratio of the corresponding standard deviations is thus given by

$$\begin{aligned}\mathcal{R}^\circ(M,L) &= \frac{\sqrt{\operatorname{Var}\left[\widehat{\mathcal{Y}}_{\mathbf{n}^\circ,\mathbf{m}}\right]}}{\sqrt{\operatorname{Var}\left[\widehat{\mathcal{Y}}_{N,M}\right]}} \\ &= \kappa^{-L/2}+\frac{\mathcal{V}_\infty^{1/2}}{M^{\beta/2}\sigma_\infty}\frac{1-\kappa^{-(1-\beta)L/2}}{1-\kappa^{-(1-\beta)/2}}.\end{aligned} \tag{4.28}$$

Note that the ratio (4.28) is independent of N. By setting the derivative of (4.28) with respect to L equal to zero we solve,

$$L^\circ := \frac{2}{\beta\ln\kappa}\ln\left[\frac{M^{\beta/2}\sigma_\infty}{\mathcal{V}_\infty^{1/2}(1-\beta)}\left(1-\kappa^{-(1-\beta)/2}\right)\right]. \tag{4.29}$$

Since $L^\circ>0$, we require

$$M>\left(\frac{\mathcal{V}_\infty^{1/2}(1-\beta)}{\sigma_\infty\left(1-\kappa^{-(1-\beta)/2}\right)}\right)^{2/\beta}. \tag{4.30}$$

It is easy to see that (4.28) attains its minimum for L° given by (4.29) and M satisfying (4.30). It then holds $\mathcal{R}^\circ(M,L^\circ)<1$, hence the multilevel estimator outperforms the standard in terms of the variance.

Remark 51 *Suppose the practitioner using the standard algorithm makes up his mind and changes his choice of N to N', connected with the number of inner simulations M. He so chooses a new budget $M\times N'$ say. Then with this new budget we can adapt the parameters accordingly, yielding the same variance reduction (4.28) with the same (4.29), as the latter are independent of N.*

4.5.1. Numerical example: American max-call

We now proceed to a numerical study of multilevel policy iteration in the context of American max-call option based on d assets. Each asset is assumed to be governed by the following SDE

$$dS_t^i=(r-\delta)S_t^i dt+\sigma S_t^i dW_t^i,\quad i=1,...,d,$$

under the risk- neutral measure, where $\left(W_t^1, ..., W_t^d\right)$ is a d-dimensional standard Brownian motion. Further, $T_0, T_1, ..., T_n$ are equidistant exercise dates between $T_0 = 0$ and T_n. For notational convenience we shall write S_j instead of S_{T_j}. The discounted cash-flow process of the option is specified by

$$Z_k = e^{-rk}\left(\max_{i=1,\ldots,d} S_k^i - K\right)^+.$$

We take the following benchmark parameter values (see Andersen and Broadie [2004])

$$r = 0.05, \quad \sigma = 0.2, \quad \delta = 0.1, \quad K = 100, \quad d = 5, \quad n = 9, \quad T_n = 3$$

and $S_0^i = 100$, $i = 1, ..., d$. For the input stopping family $(\tau_j)_{0 \le j \le T}$ we take

$$\tau_j = \inf\left\{k : j \le k < T : \, Z_k > \mathbb{E}^{\mathcal{F}_k}\left[Z_{k+1}\right]\right\} \wedge T,$$

where $\mathbb{E}^{\mathcal{F}_k}\left[Z_{k+1}\right]$ is the (discounted) value of a still-alive one period European option. The value of a European max-call option can be computed via the Johnson's formula (1987) (Johnson [1987]),

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}\left(\max_{i=1,\ldots,5} S_T^i - K\right)^+\right] \\
&= \sum_{i=1}^5 S_0^i \frac{e^{-\delta T}}{\sqrt{2\pi}} \int_{-\infty}^{d_+^i} \exp\left[-\frac{1}{2}z^2\right] \prod_{i'=1, i' \neq i}^5 \mathcal{N}\left(\frac{\ln\left(\frac{S_0^i}{S_0^{i'}}\right)}{\sigma\sqrt{k}} - z + \sigma\sqrt{T}\right) dz \\
&- Ke^{-rT} + Ke^{-rT} \prod_{i=1}^5 \left(1 - \mathcal{N}\left(d_-^i\right)\right)
\end{aligned}$$

with

$$d_+^i := \frac{\ln\left(\frac{S_0^i}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_-^i = d_+^i - \sigma\sqrt{T}.$$

For evaluating the integrals we use an adaptive Gauss-Kronrad procedure (with 31 points).

For this example we follow the approach of Section 4.5. We see that the final gain (4.28) due to the multilevel approach depends on κ as well. Our general experience is that an "optimal" κ for our method is typically larger than two. In this example we took

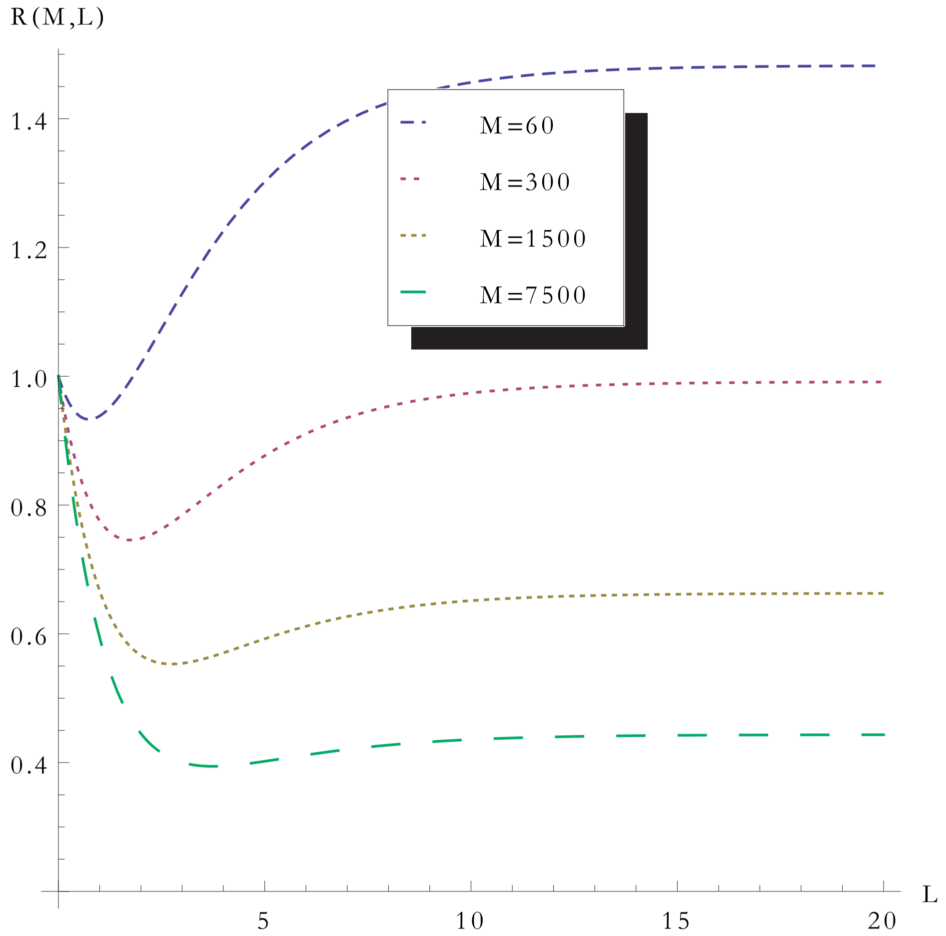

Figure 4.1.: The SD ratio function $\mathcal{R}^{\circ}(M, L)$ for different M, measuring the variance reduction due to the ML approach.

$\kappa = 5$. A pre-simulation based on 10^3 trajectories yield the following estimates,

$$\begin{aligned} \gamma = 1, \quad \beta = 0.5, \\ \operatorname{Var}\left[Z_{\widehat{\tau}_m}\right] =: \sigma^2(m) \leq \sigma_\infty^2 = 350, \\ \sqrt{m_l}\operatorname{Var}\left[Z_{\widehat{\tau}_{m_l}} - Z_{\widehat{\tau}_{m_{l-1}}}\right] \leq \mathcal{V}_\infty = 645, \end{aligned} \tag{4.31}$$

where we used antithetic sampling in (4.31). This yields Figure 4.1, where $\mathcal{R}(M, L)$ is plotted for different M as a function of L. For each particular M one may read off the optimal value of L° from this figure.

Assume, for example, that the user of the standard algorithm decides to calculate the value of the option with $M = 7500$ inner trajectories. From Figure 4.1 we see that $L = 4$

is for this M the best choice (that doesn't depend on N). For the present illustration we take $N = 1000$ and then compute n_0°, n_1° from (4.25) and (4.27), where $\mathcal{V}_\infty$ is replaced by the estimate

$$\max_{l=1,\ldots,4} \{\sqrt{m_l}\widehat{v}\,(m_l, m_{l-1})\}$$

with

$$\widehat{v}\,(m_l, m_{l-1}) := \frac{1}{n}\sum_{r=1}^{n}\left[Z^{(r)}_{\widehat{\tau}_{m_l}} - Z^{(r)}_{\widehat{\tau}_{m_{l-1}}} - \left(\overline{Z_{\widehat{\tau}_{m_l}} - Z_{\widehat{\tau}_{m_{l-1}}}}\right)\right]^2,$$

for $n = 10^3$ and the bar denoting the corresponding sample average, where antithetic variables are used in the simulation of inner trajectories. Let us further define

$$\widehat{\sigma}_m := \widehat{v}\,(m, 0) := \frac{1}{n}\sum_{r=1}^{n}\left[Z^{(r)}_{\widehat{\tau}_m} - \overline{Z_{\widehat{\tau}_m}}\right]^2$$

with $n = 10^3$ again. Table 4.1 shows the resulting values n_l°, the approximative level variances $\widehat{v}\,(m_l, m_{l-1})$, $l = 1, \ldots, 4$, as well as the option prices estimates. As can be seen from the table, the variance of the multilevel estimate $\widehat{\mathcal{Y}}_{\mathbf{n}^\circ,\mathbf{m}}$ with the "optimal" choice $L^\circ = 4$ (cf. (4.29) and Figure 4.1) is significantly smaller than the variance of the standard Monte Carlo estimate $\widehat{Y}_{1000,7500}$.

Table 4.1.: The performance of the ML estimator with the optimal choice of n_l°, $l = 0, \ldots, 4$, compared to standard policy iteration

l	n_l°	m_l	$\frac{1}{n_l^\circ}\sum_{n=1}^{n_l^\circ}\left[Z^{(n)}_{\widehat{\tau}_{m_l}} - Z^{(n)}_{\widehat{\tau}_{m_{l-1}}}\right]$	$\widehat{v}\,(m_l, m_{l-1})$
0	47368	12	25.5772	350
1	5223	60	0.0668629	53.4224
2	1847	300	−0.0623856	37.2088
3	653	1500	0.201612	15.8769
4	231	7500	−0.0319232	5.19074
			$\widehat{\mathcal{Y}}_{\mathbf{n}^\circ,\mathbf{m}} = 25.7513661$	$sd\left(\widehat{\mathcal{Y}}_{\mathbf{n}^\circ,\mathbf{m}}\right) =$ 0.2820887804
ST	$N =$ 1000	$M =$ 7500	$\widehat{Y}_{N,M} = 25.2373$	$sd\left(\widehat{Y}_{N,M}\right) =$ 0.5899033819

Concluding remarks

One may argue that the variance reduction demonstrated in the above example looks not too spectacular. In this respect we underline that this variance reduction is obtained via a pragmatic approach (Section 4.5), where detailed knowledge of the optimal allocation of the standard algorithm (in particular the precise decay of the bias) is not necessary. However, in a situation where the bias decay is additionally known (from some additional pre-computation for example), one may parameterize the multilevel algorithm following the asymptotic complexity analysis in Section 4.4, and thus end up with an (asymptotically) optimized complexity gain (4.23) that blows up when the required accuracy gets smaller and smaller.

4.6. Proofs

4.6.1. Proof of Proposition 47

Let us write $\{\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\} = \{\widehat{\tau}_{0,M} > \widehat{\tau}_0\} \cup \{\widehat{\tau}_{0,M} < \widehat{\tau}_0\}$. It then holds

$$\begin{aligned}\{\widehat{\tau}_{0,M} > \widehat{\tau}_0\} \subset \bigcup_{j=0}^{T-1} \{C_j < Z_j \leq \mathcal{C}_{j,M}\} \cap \{\widehat{\tau}_0 = j\} \\ =: \bigcup_{j=0}^{T-1} A_j^{M+} \cap \{\widehat{\tau}_0 = j\},\end{aligned}$$

and similarly,

$$\begin{aligned}\{\widehat{\tau}_{0,M} < \widehat{\tau}_0\} \subset \bigcup_{j=0}^{T-1} \{C_j \geq Z_j > \mathcal{C}_{j,M}\} \cap \{\widehat{\tau}_0 = j\} \\ =: \bigcup_{j=0}^{T-1} A_j^{M-} \cap \{\widehat{\tau}_0 = j\}.\end{aligned}$$

So we have

$$\mathbb{P}\left(\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\right) \leq \sum_{j=0}^{T-1} \mathbb{P}\left(A_j^{M+} \cup A_j^{M-}\right).$$

By the conditional version of the Bernstein inequality, namely for $t > 0$

$$\mathbb{P}\left(t < \sum_{i=1}^{M} X_i\right) \leq \exp\left(-\frac{1/2t^2}{\sum_{i=1}^{M} \mathbb{E}\left[X_i^2\right] + 1/3Bt}\right),$$

for independent variables $X_1, ..., X_M$ with zero mean, where $|X_i| \leq B$ $a.s.$ for all i, we have

$$\begin{aligned}
\mathbb{P}_{\mathcal{F}_T}\left(A_j^{M+}\right) &= \mathbb{P}_{X_j}\left(0 < Z_j - C_j \leq \frac{1}{M}\sum_{m=1}^{M}\left(Z_{\tau_{j+1}^{(m)}}(X_{\tau_{j+1}^{(m)}}^{j,X_j(m)}) - C_j\right)\right) \\
&\leq \mathbf{1}_{\left\{|Z_j-C_j|\leq M^{-1/2}\right\}} + \sum_{k=1}^{\infty}\mathbf{1}_{\left\{2^{k-1}M^{-1/2}<|Z_j-C_j|\leq 2^k M^{-1/2}\right\}}\cdot \\
&\cdot \mathbb{P}_{X_j}\left(2^{k-1}M^{-1/2} < \frac{1}{M}\sum_{m=1}^{M}\left(Z_{\tau_{j+1}^{(m)}}(X_{\tau_{j+1}^{(m)}}^{j,X_j,(m)}) - C_j\right)\right) \\
&\leq \mathbf{1}_{\left\{|Z_j-C_j|\leq M^{-1/2}\right\}} + \sum_{k=1}^{\infty}\mathbf{1}_{\left\{2^{k-1}M^{-1/2}<|Z_j-C_j|\leq 2^k M^{-1/2}\right\}}\cdot \\
&\cdot \exp\left[-\frac{2^{2k-3}M}{MB^2 + B2^{k-1}M^{1/2}/3}\right] \\
&\leq \mathbf{1}_{\left\{|Z_j-C_j|\leq M^{-1/2}\right\}} + \sum_{k=1}^{\infty}\mathbf{1}_{\left\{|Z_j-C_j|\leq 2^k M^{-1/2}\right\}}\cdot \\
&\cdot \exp\left[-\frac{2^{2k-2}}{B^2 + B2^{k-1}/3}\right].
\end{aligned}$$

So by assumption (4.9),

$$\begin{aligned}
\mathbb{P}\left(A_j^{M+}\right) &\leq DM^{-\alpha/2} + D\sum_{k=1}^{\infty}2^{\alpha k}M^{-\alpha/2}\exp\left[-\frac{2^{2k-2}}{B^2 + B2^{k-1}M^{-1/2}/3}\right] \\
&\leq B_1 M^{-\alpha/2}
\end{aligned}$$

for B_1 depending on B, and α. After obtaining a similar estimate $\mathbb{P}\left(A_j^{M-}\right) \leq B_2 M^{-\alpha/2}$, we finally conclude that

$$\mathbb{P}\left(\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\right) \leq M^{-\alpha/2} T \max(B_1, B_2) =: D_1 M^{-\alpha/2}.$$

4.6.2. Proof of Theorem 49

Define $\widehat{\tau}_M := \widehat{\tau}_{0,M}$, $\widehat{\tau} := \widehat{\tau}_0$, and use induction to the number of exercise dates T. For $T = 0$ the statement is trivially fulfilled. Suppose it is shown that

$$\mathbb{E}\left(Z_{\widehat{\tau}_M} - Z_{\widehat{\tau}}\right) = O(\frac{1}{M})$$

for T exercise dates. Now consider the cash-flow process $Z_0, ..., Z_{T+1}$. Note that the filtration $(\mathcal{F}_j)$ is generated by the outer trajectories. Note, since $T+1$ is the last exercise date, the event $\{\widehat{\tau} = T+1\} = \Omega \backslash \{\widehat{\tau} \leq T\}$ is $\mathcal{F}_T$-measurable. Further, the event $\{\widehat{\tau}_M = T+1\} = \Omega \backslash \{\widehat{\tau}_M \leq T\}$ is measurable with respect to the information generated by the inner simulated trajectories starting from an outer trajectory at time T, and so, in particular, does **not** depend on the information generated by the the outer trajectories from T until $T+1$. That is, we have

$$\mathbb{E}^{\mathcal{F}_{T+1}}\left[\left(1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right)\right]=\mathbb{E}^{\mathcal{F}_T}\left[\left(1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right)\right]$$

and so

$$\begin{aligned}\mathbb{E}\left[Z_{T+1}\left(1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right)\right]=\mathbb{E}\left[Z_{T+1}\mathbb{E}^{\mathcal{F}_{T+1}}\left(1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right)\right]\\ \mathbb{E}\left[Z_{T+1}\mathbb{E}^{\mathcal{F}_T}\left[\left(1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right)\right]\right].\end{aligned}\tag{4.32}$$

By (4.32) and applying the induction hypothesis to the modified cash-flow $Z_j 1_{j\le T}$, it then follows that

$$\begin{aligned}\left|\mathbb{E}\left(Z_{\widehat{\tau}_M}-Z_{\widehat{\tau}}\right)\right|&=\left|\mathbb{E}\left(Z_{\widehat{\tau}_M}1_{\widehat{\tau}_M\le T}+Z_{T+1}1_{\widehat{\tau}_M=T+1}-Z_{\widehat{\tau}}1_{\widehat{\tau}\le T}-Z_{T+1}1_{\widehat{\tau}=T+1}\right)\right|\\&=\left|\mathbb{E}\left(Z_{\widehat{\tau}_M}1_{\widehat{\tau}_M\le T}-Z_{\widehat{\tau}}1_{\widehat{\tau}\le T}\right)+\mathbb{E}\left(Z_{T+1}\left(1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right)\right)\right|\\&\le O(\frac{1}{M})+\left|\mathbb{E}\left(Z_{T+1}\mathbb{E}^{\mathcal{F}_T}\left(1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right)\right)\right|.\end{aligned}\tag{4.33}$$

Let us estimate the second term $\mathbb{E}\left[Z_{T+1}\mathbb{E}^{\mathcal{F}_T}\left[1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right]\right]$. Denote $\varepsilon_{M,j}=1_{Z_j\le\mathcal{C}_{j,M}}-1_{Z_j\le C_j}$ for $j=0,...,T$, and $\overline{\varepsilon}_{M,j}=\mathbb{E}^{\mathcal{F}_j}\left[1_{Z_j\le\mathcal{C}_{j,M}}-1_{Z_j\le C_j}\right]$. Then by the identity ($i_0:=+\infty$)

$$\prod_{i=1}^n a_i-\prod_{i=1}^n b_i=\sum_{l=1}^n\sum_{i_l<i_{l-1}<\cdots<i_0}\prod_{r=1}^l(a_{i_r}-b_{i_r})\cdot\prod_{j\ne i_l,j\ne i_{l-1},\dots,j\ne i_1}b_j$$

it holds

$$\mathbb{E}^{\mathcal{F}_T}\left[1_{\widehat{\tau}_M=T+1}-1_{\widehat{\tau}=T+1}\right]=\mathbb{E}^{\mathcal{F}_T}\left[\prod_{j=0}^T 1_{Z_j\le\mathcal{C}_{j,M}}-\prod_{j=0}^T 1_{Z_j\le C_j}\right]=\mathcal{R}_1+\mathcal{R}_2,$$

where

$$\mathcal{R}_1=\mathbb{E}^{\mathcal{F}_T}\left[\sum_{j=0}^T\varepsilon_{M,j}\prod_{i\ne j}1_{Z_i\le C_i}\right]=\sum_{j=0}^T\overline{\varepsilon}_{M,j}\prod_{i\ne j}1_{Z_i\le C_i}$$

and

$$
\begin{aligned}
\mathcal{R}_2 &= \mathbb{E}^{\mathcal{F}_T}\left[\sum_{j_2<j_1} \varepsilon_{M,j_1}\varepsilon_{M,j_2} \prod_{i\neq j_1,j_2} 1_{Z_i\leq C_i}\right] \\
&+ \mathbb{E}^{\mathcal{F}_T}\left[\sum_{j_3<j_2<j_1} \varepsilon_{M,j_1}\varepsilon_{M,j_2}\varepsilon_{M,j_3} \prod_{i\neq j_1,j_2,j_3} 1_{Z_i\leq C_i}\right] + \ldots \\
&= \sum_{j_2<j_1} \overline{\varepsilon}_{M,j_1}\overline{\varepsilon}_{M,j_2} \prod_{i\neq j_1,j_2} 1_{Z_i\leq C_i} \\
&+ \sum_{j_3<j_2<j_1} \overline{\varepsilon}_{M,j_1}\overline{\varepsilon}_{M,j_2}\overline{\varepsilon}_{M,j_3} \prod_{i\neq j_1,j_2,j_3} 1_{Z_i\leq C_i} + \ldots
\end{aligned}
$$

were we note that conditional $\mathcal{F}_T$ the $\varepsilon_{M,j}$ are independent. It is easy to show that

$$
\overline{\varepsilon}_{M,j} = O_P\left(\frac{1}{\sqrt{M}}\right), \quad \text{hence} \quad \mathbb{E}[Z_{T+1}\mathcal{R}_2] = O\left(\frac{1}{M}\right)..
$$

Let us write

$$
\begin{aligned}
\mathbb{E}[Z_{T+1}\mathcal{R}_1] &= \sum_{j=0}^{T} \mathbb{E}\left[Z_{T+1}\overline{\varepsilon}_{M,j} \prod_{i\neq j} 1_{Z_i\leq C_i}\right] \\
&= \sum_{j=0}^{T} \mathbb{E}\left[\overline{\varepsilon}_{M,j}\mathbb{E}^{\mathcal{F}_j}\left[Z_{T+1} \prod_{i\neq j} 1_{Z_i\leq C_i}\right]\right] \\
&=: \sum_{j=0}^{T} \mathbb{E}\left[\overline{\varepsilon}_{M,j} W_j\right]..
\end{aligned}
$$

By assumption, $Z_j = Z_j(X_j)$, $j = 0, \ldots, T$. Let us set

$$
f_j(x) := Z_j(x) - \mathbb{E}[Z_{\tau_{j+1}}(X_{\tau_{j+1}})|X_j = x] = Z_j(x) - C_j(x)
$$

and consider for fixed j,

$$
\mathcal{C}_{j,M} - C_j = \frac{1}{M}\sum_{m=1}^{M}\left(Z_{\tau_{j+1}^{(m)}}(X_{\tau_{j+1}^{(m)}}^{j,x,(m)}) - C_j(x)\right) =: \sigma_j(x)\frac{\Delta_{j,M}(x)}{\sqrt{M}}
$$

where σ_j is defined in **(iii)**, and denote by $p_{j,M}(\cdot;x)$ the conditional density of the random variable $\Delta_{j,M}(x)$ given $X_j = x$. Then

$$\begin{aligned}
\mathbb{E}[Z_{T+1}\mathcal{R}_1] &= \sum_{j=0}^{T}\mathbb{E}\left[W_j\mathbb{E}^{\mathcal{F}_j}\left[1_{Z_j\le\mathcal{C}_{j,M}} - 1_{Z_j\le C_j}\right]\right]\\
&= \sum_{j=0}^{T}\mathbb{E}\left[W_j\mathbb{E}^{\mathcal{F}_j}\left[1_{\left\{f_j(X_j)\le\mathcal{C}_{j,M}-C_j\right\}} - 1_{f_j(X_j)\le 0}\right]\right]\\
&= \sum_{j=0}^{T}\mathbb{E}\left[W_j\mathbb{E}^{\mathcal{F}_j}\left[1_{\left\{f_j(X_j)\le\sigma_j(x)\frac{\Delta_{j,M}(X_j)}{\sqrt{M}}\right\}} - 1_{f_j(X_j)\le 0}\right]\right]\\
&= \sum_{j=0}^{T}\mathbb{E}\left[W_j\int p_{j,M}(z;X_j)\left(1_{\left\{f_j(X_j)\le\sigma_j(X_j)\frac{z}{\sqrt{M}}\right\}} - 1_{f_j(X_j)\le 0}\right)dz\right]\\
&= \sum_{j=0}^{T}\mathbb{E}\left[1_{f_j(X_j)>0}W_j\int p_{j,M}(z;X_j)\left(1_{\left\{f_j(X_j)\le\sigma_j(X_j)\frac{z}{\sqrt{M}}\right\}} - 1_{f_j(X_j)\le 0}\right)dz\right]\\
&+ \sum_{j=0}^{T}\mathbb{E}\left[1_{f_j(X_j)\le 0}W_j\int p_{j,M}(z;X_j)\left(1_{\left\{f_j(X_j)\le\sigma_j(X_j)\frac{z}{\sqrt{M}}\right\}} - 1_{f_j(X_j)\le 0}\right)dz\right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{T}\mathbb{E}\left[W_j\int p_{j,M}(z;X_j)1_{\left\{f_j(X_j)\le\sigma_j(X_j)\frac{z}{\sqrt{M}}\right\}}dz\right]\\
&+ \sum_{j=0}^{T}\mathbb{E}\left[1_{f_j(X_j)\le 0}W_j\int p_{j,M}(z;X_j)\left(1_{\left\{f_j(X_j)\le\sigma_j(X_j)\frac{z}{\sqrt{M}}\le 0\right\}} - 1\right)dz\right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{T}\mathbb{E}\left[W_j\int p_{j,M}(z;X_j)1_{\left\{0<f_j(X_j)\le\sigma_j(X_j)\frac{z}{\sqrt{M}}\right\}}dz\right]\\
&- \sum_{j=0}^{T}\mathbb{E}\left[W_j\int p_{j,M}(z;X_j)1_{\left\{\sigma_j(X_j)\frac{z}{\sqrt{M}}<f_j(X_j)\le 0\right\}}dz\right]\\
&= \sum_{j=0}^{T}(I)_j - \sum_{j=0}^{T}(II)_j
\end{aligned}$$

Note that

$$
\begin{aligned}
W_j &= \prod_{i<j} 1_{Z_i \le C_i} \mathbb{E}^{\mathcal{F}_j} \left[Z_{T+1} \prod_{i>j} 1_{Z_i \le C_i} \right] \\
&=: \prod_{i<j} 1_{Z_i \le C_i} V_j(X_j),
\end{aligned}
$$

so

$$
(I)_j = \mathbb{E} \left[\prod_{i<j} 1_{Z_i \le C_i} \mathbb{E}^{\mathcal{F}_j} V_j(X_j) \int p_{j,M}(z; X_j) 1_{\left\{0 < f_j(X_j) \le \sigma_j(X_j) \frac{z}{\sqrt{M}}\right\}} dz \right],
$$

Consider

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_j} V_j(X_j) \int p_{j,M}(z; X_j) dz 1_{\left\{0 < f_j(X_j) \le \sigma_j(X_j) \frac{z}{\sqrt{M}}\right\}} \\
&=: \int \mathfrak{p}_j\left(x; X_{j-1}\right) V_j(x) dx \int p_{j,M}(z; x) 1_{\left\{0 < f_j(x) \le \sigma_j(x) \frac{z}{\sqrt{M}}\right\}} dz \\
&= \int dz \int p_{j,M}(z; x) \mathfrak{p}_j\left(x; X_{j-1}\right) V_j(x) 1_{\left\{0 < f_j(x) \le \sigma_j(x) \frac{z}{\sqrt{M}}\right\}} dx
\end{aligned}
$$

Similarly,

$$
(II)_j = \mathbb{E} \left[\prod_{i<j} 1_{Z_i \le C_i} \mathbb{E}^{\mathcal{F}_j} \left(V_j(X_j) \int p_{j,M}(z; X_j) 1_{\left\{\sigma_j(X_j) \frac{z}{\sqrt{M}} < f_j(X_j) \le 0\right\}} \right) dz \right]
$$

where

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_j} V_j(X_j) \int p_{j,M}(z; X_j) 1_{\left\{\sigma_j(X_j) \frac{z}{\sqrt{M}} < f_j(X_j) \le 0\right\}} dz \\
&= \int dz \int p_{j,M}(z; x) \mathfrak{p}_j\left(x; X_{j-1}\right) V_j(x) 1_{\left\{\sigma_j(x) \frac{z}{\sqrt{M}} < f_j(x) \le 0\right\}} dx,
\end{aligned}
$$

yielding

$$
\begin{aligned}
(I)_j - (II)_j = &\mathbb{E} \left[\prod_{i<j} 1_{Z_i \le C_i} \int dz \int p_{j,M}(z; x) \mathfrak{p}_j\left(x; X_{j-1}\right) V_j(x) 1_{\left\{0 < f_j(x) \le \sigma_j(x) \frac{z}{\sqrt{M}}\right\}} dx \right] \\
&- \mathbb{E} \left[\prod_{i<j} 1_{Z_i \le C_i} \int dz \int p_{j,M}(z; x) \mathfrak{p}_j\left(x; X_{j-1}\right) V_j(x) 1_{\left\{\sigma_j(x) \frac{z}{\sqrt{M}} < f_j(x) \le 0\right\}} dx \right]
\end{aligned}
$$

$$= \int dz \int p_{j,M}(z;x) V_j(x) \psi_j(x) 1_{\left\{0 < f_j(x) \leq \sigma_j(x) \frac{z}{\sqrt{M}}\right\}} dx$$
$$- \int dz \int p_{j,M}(z;x) V_j(x) \psi_j(x) 1_{\left\{\sigma_j(x) \frac{z}{\sqrt{M}} < f_j(x) \leq 0\right\}} dx$$
$$=: (*)_1 - (*)_2,$$

where

$$\psi_j(x) := \mathbb{E}\left[\prod_{i<j} 1_{Z_i \leq C_i} \mathfrak{p}_j\left(x; X_{j-1}\right)\right]$$

Now we assume that $\sigma_j(x)$ is, uniformly in x and j, bounded and bounded away from zero, and that

$$p_{j,M}(z;x) = \phi\left(z\right)\left(1 + \frac{D_{j,M}(z;x)}{\sqrt{M}}\right)$$

with ϕ being the standard normal density and with $D_{j,M}$ satisfying for all x and M the normalization condition

$$\int \phi\left(w\right) D_{j,M}(w;x) dw = 0,$$

and the growth bound

$$D^x_{j,M}(w) = O(e^{aw^2/2}) \text{ for some } a < 1 \text{ uniformly in } j, M \text{ and } x. \tag{4.34}$$

For example, (4.34) is fulfilled if the cash-flow $Z_j(x)$ is uniformly bounded in j and x (see Appendix). We then have

$$(*)_1 = \int dz \int \phi\left(z\right) V_j(x) \psi_j(x) 1_{\left\{0 < f_j(x)/\sigma_j(x) \leq \frac{z}{\sqrt{M}}\right\}} dx$$
$$+ \int dz \int \phi\left(z\right) \frac{D_{j,M}(z;x)}{\sqrt{M}} V_j(x) \psi_j(x) 1_{\left\{0 < f_j(x)/\sigma_j(x) \leq \frac{z}{\sqrt{M}}\right\}} dx$$
$$=: (*)_{1a} + (*)_{1b}$$

Let $\xi_j(dy)$ be the image of the measure

$$V_j(x) \psi_j(x) dx$$

under the map

$$x \to \frac{f_j(x)}{\sigma_j(x)}$$

Then,

$$(*)_{1a} = \int dz \phi\left(z\right) 1_{z>0} \xi_j((0, \frac{z}{\sqrt{M}}])$$
$$= \sqrt{M} \int 1_{t>0} dt\, \phi\left(t\sqrt{M}\right) \xi_j((0,t]).$$

Since $\xi_j((0,0]) = 0$ and the fact $V_j > 0$ is infinitely differentiable, we have due to assumptions (i)-(iv) that ξ has a density $g(0)$ in $t = 0$, and that

$$\xi_j((-t,0]) = tg(0) + O(t^2) = \xi_j((0,t]), \quad t > 0. \tag{4.35}$$

Then by following the standard Laplace method for integrals (e.g. see de Bruijn [1981]) we get

$$\begin{aligned}(*)_{1a} &= \frac{\sqrt{M}}{2\pi} \int 1_{t>0} dt e^{-\ Mt^2/2} \xi_j((0,t]) \\ &= d_j M^{-1/2} + O(M^{-1}).\end{aligned}$$

Further we have for some constant C

$$\begin{aligned}|(*)_{1b}| &\leq \frac{C}{\sqrt{M}} \int dz \int \phi\left(z\right) e^{az^2/2} V_j(x) \psi_j(x) 1_{\left\{0 < \frac{f_j(x)}{\sigma_j(x)} \leq \frac{w}{\sqrt{M}}\right\}} dx \\ &= \frac{C}{\sqrt{2\pi M}} \int e^{-\frac{1}{2}(1-a)z^2} \xi_j((0, \frac{z}{\sqrt{M}}]) dz \\ &= \frac{C}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-a)t^2 M} \xi_j((0,t]) dt = O(M^{-1}).\end{aligned}$$

Due to (4.35) we get in the same way $(*)_2 = (*)_{2a} + (*)_{2b}$,

$$\begin{aligned}(*)_{2a} &= \sqrt{M} \int 1_{t<0} dt \phi\left(t\sqrt{M}\right) \xi_j((t,0]) = \int 1_{t>0} dt \phi\left(t\sqrt{M}\right) \xi_j((-t,0]) \\ &= d_j M^{-1/2} + O(M^{-1})\end{aligned}$$

and $(*)_{2b} = O(M^{-1})$. Gathering all together we obtain $(*)_1 - (*)_2 = O(M^{-1})$, hence $(I)_j - (II)_j = O(M^{-1})$ for all j, and we so finally arrive at

$$\mathbb{E}\left[Z_{T+1}(1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau} = T+1})\right] = O(M^{-1}).$$

4.6.3. Proof of Theorem 50

First, we analyze the variance of the estimator $\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}}$, that is given by

$$\begin{aligned}\mathrm{Var}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}}\right] &= \frac{1}{n_0} \mathrm{Var}\left[Z_{\widehat{\tau}_{m_0}}\right] + \sum_{l=1}^{L} \frac{1}{n_l} \mathrm{Var}\left[Z_{\widehat{\tau}_{m_l}} - Z_{\widehat{\tau}_{m_{l-1}}}\right] \\ &\leq \frac{\sigma_\infty^2}{n_0} + \frac{\mathcal{V}_\infty \kappa^{-\beta}}{n_1 m_0^\beta} \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1},\end{aligned} \tag{4.36}$$

cf. (4.12) and (4.17). Let us now minimize the complexity (4.18) over the parameters n_0 and n_1, for given L, m_0 and accuracy ϵ, that is (cf. (4.12)),

$$\left(\frac{\mu_\infty}{m_0^\gamma \kappa^{\gamma L}}\right)^2 + \frac{\sigma_\infty^2}{n_0} + \frac{\mathcal{V}_\infty \kappa^{-\beta}}{n_1 m_0^\beta} \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} = \epsilon^2.$$

We thus have to choose L such that $\frac{\mu_\infty}{m_0^\gamma \kappa^{\gamma L}} < \epsilon$, i.e.,

$$L > \gamma^{-1} \frac{\ln \epsilon^{-1} + \ln\left(\mu_\infty / m_0^\gamma\right)}{\ln \kappa}. \tag{4.37}$$

With a Lagrangian optimization we find

$$n_0^*(L, m_0, \epsilon) = \frac{\sigma_\infty^2 + \sigma_\infty \mathcal{V}_\infty^{1/2} m_0^{-\beta/2} \frac{\kappa^{L(1-\beta)/2}-1}{1-\kappa^{-(1-\beta)/2}}}{\epsilon^2 - \left(\frac{\mu_\infty}{m_0^\gamma \kappa^{L\gamma}}\right)^2}, \tag{4.38}$$

$$n_1^*(L, m_0, \epsilon) = n_0^*(L, m_0, \epsilon)\, \sigma_\infty^{-1} \kappa^{-(1+\beta)/2} \mathcal{V}_\infty^{1/2} m_0^{-\beta/2}. \tag{4.39}$$

This results in a complexity (see (4.18))

$$\begin{aligned} \mathcal{C}_{ML}(n_0^*, n_1^*, L, m_0, \epsilon) &:= n_0^*(L, m_0, \epsilon)\, m_0 + n_1^*(L, m_0, \epsilon)\, m_0 \kappa \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} \\ &= \frac{\left(\sigma_\infty m_0^{\beta/2} + \sqrt{\mathcal{V}_\infty} \frac{\kappa^{L(1-\beta)/2}-1}{\kappa^{(1-\beta)/2}-1} \kappa^{(1-\beta)/2}\right)^2 m_0^{1-\beta}}{\epsilon^2 - \left(\frac{\mu_\infty}{m_0^\gamma \kappa^{L\gamma}}\right)^2}. \end{aligned} \tag{4.40}$$

Next we are going to optimize over L. To this end we differentiate (4.40) to L and set the derivative equal to zero, which yields,

$$\begin{aligned} \epsilon^2 \kappa^{2L\gamma} &= \frac{\mu_\infty^2}{m_0^{2\gamma}} \left(1 + 2\gamma / (1-\beta)\right) \\ &\quad + \frac{2\gamma}{1-\beta} \frac{\mu_\infty^2}{m_0^{2\gamma}} \left(1 + \sigma_\infty m_0^{\beta/2} \mathcal{V}_\infty^{-1/2} \left(1 - \kappa^{-(1-\beta)/2}\right)\right) \kappa^{-L(1-\beta)/2} \\ &=: p + q\kappa^{-L(1-\beta)/2}, \quad \text{with} \end{aligned} \tag{4.41}$$

$$L = \frac{\ln \epsilon^{-1}}{\gamma \ln \kappa} + \frac{\ln p}{2\gamma \ln \kappa} + \frac{\ln\left(1 + q\kappa^{-L(1-\beta)/2}/p\right)}{2\gamma \ln \kappa}. \tag{4.42}$$

From (4.41) we see that there is at most one solution in L, and since $\beta < 1$ we see from (4.42) that $L \to \infty$ as $\epsilon \downarrow 0$. So we may write

$$L = \frac{\ln \epsilon^{-1}}{\gamma \ln \kappa} + \frac{\ln p}{2\gamma \ln \kappa} + O\left(\kappa^{-L(1-\beta)/2}\right), \quad \epsilon \downarrow 0. \tag{4.43}$$

Due to (4.43) we have that

$$L = \frac{\ln \epsilon^{-1}}{\gamma \ln \kappa} + O(1), \quad \epsilon \downarrow 0, \tag{4.44}$$

hence by iterating (4.43) with (4.44) once, we obtain the asymptotic solution

$$L^* := \frac{\ln \epsilon^{-1}}{\gamma \ln \kappa} + \frac{\ln p}{2\gamma \ln \kappa} + O\left(\epsilon^{(1-\beta)/(2\gamma)}\right), \quad \epsilon \downarrow 0, \tag{4.45}$$

that obviously satisfies (4.37) for ϵ small enough. We now are ready to prove the following asymptotic complexity theorem. Due to (4.45) it holds for $a > 0$,

$$\kappa^{aL^*} = p^{a/(2\gamma)}\epsilon^{-a/\gamma}\left(1 + O\left(\epsilon^{(1-\beta)/(2\gamma)}\right)\right), \quad \text{hence}$$

$$\kappa^{L^*(1-\beta)/2} = p^{(1-\beta)/(4\gamma)}\epsilon^{-(1-\beta)/(2\gamma)} + O(1) \quad \text{and} \tag{4.46}$$

$$\kappa^{\gamma L^*} = p^{1/2}\epsilon^{-1}\left(1 + O\left(\epsilon^{(1-\beta)/(2\gamma)}\right)\right). \tag{4.47}$$

So by inserting (4.46), (4.47) with (4.41) in (4.40) we get after elementary algebraic and asymptotic manipulations (4.19). By inserting (4.46), (4.47) with (4.41) in (4.38) and (4.39) respectively we get in the same way (4.20) and (4.21), respectively. Finally, combining (4.41) and (4.45) yields (4.22).

4.7. Outlook

We considered here a Howard policy iteration, resulting in a window parameter of one, while looking at one iteration step. Issues of concern for future research are given by an extension of the Howard policy iteration, taking into account either a window parameter greater one, as considered in Kolodko and Schoenmakers [2006] or multiple iterations. A combination of both might be considered the final goal.

5. Robust Optimal Stopping

We study here the optimal stopping problem in the presence of model uncertainty (ambiguity). We develop a method to practically solve this problem in a general setting, allowing for general time-consistent ambiguity averse preferences and general payoff processes driven by jump-diffusions. Our method consists of four steps. First, we construct a suitable Doob martingale associated with the solution to the optimal stopping problem using backward stochastic calculus. Within this step we construct the continuation value and use it as a exercise criteria, thus extracting a lower bound. Second, we employ this martingale to construct an approximated upper bound to the solution using duality. Third, we introduce backward-forward simulation to obtain a genuine upper bound to the solution, which converges to the true solution asymptotically. Fourth we will use the continuation value to extract stopping times. These will be used to calculate a lower by using BSDEs with random terminal time. We analyze the asymptotic behavior and convergence properties of our method. The method is illustrated in various numerical examples.

5.1. Problem Description

5.1.1. Setting, Rewards and Preferences

We work under a similar setup as in Section 1.1, this time assuming we are given a constant interest rate given by r as discussed at the end of that section. As this Chapter is dedicate to the description of the algorithm we postpone the proofs of our theoretical results to an own Section, Section 5.5 for readability.

Consider a decision-maker (economic agent or firm) who has to decide at what time to stop (or exercise) a certain action in order to maximize his future uncertain (sequence of) rewards. For the dynamics of the rewards, we assume a continuous-time jump-diffusion setting with ambiguity. Ambiguity, which is also referred to as model uncertainty, means that the "true" probabilistic model is unknown to the decision-maker. Note that it seems to be a strong assumption stating that the trends and patterns of the economic shocks in the future will be similar to the trends and patterns displayed by the economic shocks of the past. Therefore, jumps and ambiguity seems to go together rather well. Formally, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{P})$ and assume that the probability space is equipped with two independent processes, which will serve as our stochastic drivers:

(i) A standard d-dimensional Brownian motion $W = (W_1, \ldots, W_d)^T$.

(ii) A standard k-dimensional Poisson process $N = (N_1, \ldots, N_k)^T$ with intensities $\lambda^{\mathbb{P}} = \left(\lambda_1^{\mathbb{P}}, \ldots, \lambda_k^{\mathbb{P}}\right)^T$.

Standard in this case means that the components are assumed to be independent, and, in the case of W, to have zero mean and unit variance. We denote the vector of compensated Poisson processes by $\tilde{N} = \left(\tilde{N}_1, \ldots, \tilde{N}_k\right)^T$, where

$$\tilde{N}_i(t) = N_i(t) - \lambda_i^{\mathbb{P}} t, \qquad i = 1, \ldots, k.$$

We assume that these stochastic drivers generate an n-dimensional adapted Markov process $(X(t))_{0 \leq t \leq T}$ satisfying the strong Markov property. The process X is exogenous and may represent a production process, a capacity process, a stream of net cash flows, or a price process of e.g., a collection of risky assets.

The decision-maker chooses a stopping time τ taking values between time 0 and a fixed maturity time $T < \infty$. We assume that if the decision-maker exercises at time $\tau = t_i$, he receives the reward

$$H(t_i) = \Pi(t_i, X(t_i)) + \sum_{j=i}^{L} h(t_j, X(t_j)), \qquad t_i \in \{0 = t_0, t_1, \ldots, t_L = T\}, \tag{5.1}$$

for functions Π and h mapping from $\{0 = t_0, t_1, \ldots, t_L = T\} \times \mathbb{R}^n$ to $\mathbb{R}$. Furthermore, we assume that $h(t_j, X(t_j)) \in L^2$ for all $j = 0, \ldots, L$. Standard examples that take the form (5.1) include:

(a) The optimal entrance problem: In this case, typically $\Pi(t,x) = -\frac{k}{(1+r)^t}$ for a fixed irreversible cost k that depreciates at rate r, and $h(t,x) = \frac{h(x)-c}{(1+r)^t}$, which measures the present value of the payoff or the production per time unit, $h(x)$, after entering the market, minus the running costs, c. Often times $h(x)$ is simply taken to be equal to x. Of course, the fixed costs may also depend on the state of the economy at time t, $X(t)$.

(b) The optimal (simple) reward problem: In this case, $h = 0$ and $\Pi(t,x)$ is the (simple) reward function of exercising at time t. This problem appears abundantly in the American option pricing literature, with $X(t)$ a vector of risky asset values at time t.

For further details on these and other examples, see the references provided in the Introduction.

In standard optimal stopping problems, the decision-maker maximizes the expected reward under a given model $\mathbb{P}$:

$$\max_{\tau \in \mathcal{T}} \mathbb{E}\left[H(\tau)\right],$$

where $\mathcal{T} = \{0 = t_0 < t_1 < \ldots < t_L = T\}$ is the set of possible exercise dates. Usually this problem can then be solved either explicitly or dynamically by a backward recursion. Specifying the model $\mathbb{P}$ in our setting implies specifying the distribution of the whole path $(X(t))_{0 \leq t \leq T}$. In reality, however, the probabilities with which future rewards are received are often times subject to model uncertainty. This is for instance the case if data are scarce, statistical models underlying the evaluation procedures are inadequate, or, for evaluations which rely on extrapolating past trends, patterns of the future can be different from those of the past. Similarly in financial decision making (as in the American option example) investors need to cope with markets which are inherently incomplete meaning in particular that the equivalent martingale measure with which $(H(t))$ has to be priced is not unique. Therefore, it is appealing to consider instead a robust decision criterion, which induces that the optimal stopping strategy accounts for a whole class of probabilistic models and not just a single model. In other words, an agent would be willing to pay an additional ambiguity premium if he could be sure that his estimate of $\mathbb{P}$ is the correct one. Different approaches to decision-making under ambiguity have emerged in the literature. Among the most popular approaches is multiple priors (Gilboa and Schmeidler [1989]) and variational preferences (Maccheroni, Marinacci and Rustichini [2006]). With linear utility, these decision criteria correspond to coherent (Artzner, Delbaen, Eber and Heath [1999]) and convex measures of risk (Föllmer and Schied [2002]). Henceforth, we postulate that the decision-maker adopts a convex measure of risk and evaluates his future reward according to

$$U(H(\tau)) = \inf_{\mathbb{Q}\in\mathcal{Q}} \left\{\mathbb{E}_{\mathbb{Q}}[H(\tau)] + c(\mathbb{Q})\right\}, \tag{5.2}$$

with $\mathcal{Q} = \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}$ and $c : \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$. (We call $\mathbb{Q}$ equivalent to $\mathbb{P}$ and write $\mathbb{Q} \sim \mathbb{P}$ if events that have probability zero under $\mathbb{P}$ still have probability zero under $\mathbb{Q}$ and vice versa.) For our purposes, we have to consider the dynamic version of (5.2), given by

$$U(t, H(\tau)) = \inf_{\mathbb{Q}\in\mathcal{Q}} \left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}[H(\tau)] + c(t,\mathbb{Q})\right\}, \tag{5.3}$$

in which $c(t,\mathbb{Q})$ reflects the esteemed plausibility of the model $\mathbb{Q}$ given the information up to time t. In (5.3), and in the rest of this Chapter, we define for notational convenience sup := ess.sup and inf := ess.inf. The optimal stopping problem at time t_i is then given by

$$V^*(t_i) = \sup_{\tau\in\mathcal{T}_i} U(t_i, H(\tau)) = \sup_{\tau\in\mathcal{T}_i} \inf_{\mathbb{Q}\in M} \left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_{t_i}}[H(\tau)] + c(t_i,\mathbb{Q})\right\}, \tag{5.4}$$

with $\mathcal{T}_i := \{t_j \geq t_i | t_j \in \mathcal{T}\}$.

5.1.2. Time-Consistency, Dynamic Programming and Assumptions

We now consider the question of which class of plausibility indices (penalty functions) to employ in (5.3)–(5.4). To this end, we first recall the notion of time-consistency in dynamic choice problems under uncertainty. We say that a dynamic evaluation $(U(t,H))_{0\leq t\leq T}$ is time-consistent if

$$U(t, H_2) \geq U(t, H_1) \qquad \Rightarrow \qquad U(s, H_2) \geq U(s, H_1), \qquad t \geq s.$$

This means that if, in each state of nature at time t, the reward H_2 is preferred over the reward H_1, then H_1 should also have been preferred over H_2 prior to time t. It turns out that requiring time-consistency of U is equivalent to requiring that U satisfies a dynamic programming principle, which, in turn, is equivalent in our setting to the penalty function associated with U taking a certain form, specified later.

Next, we explain what a change of measure from $\mathbb{P}$ to $\mathbb{Q}$ implies in our setting. If $\mathbb{Q} \sim \mathbb{P}$, we denote by $D(t)$ the Radon-Nikodym derivative $D(t) = \mathbb{E}^{\mathcal{F}_t}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. In our jump-diffusion setting it is known that, for every model $\mathbb{Q} \sim \mathbb{P}$, there exist a predictable, $\mathbb{R}^d$-valued, stochastic drift q and a positive, predictable, $\mathbb{R}^k$-valued process λ such that the Radon-Nikodym derivative can be written as

$$\begin{aligned} D(t) \;=\; & \exp\left\{\int_0^t q(s)\,dW(s) + \int_0^t \log\left(\frac{\lambda(s)}{\lambda^{\mathbb{P}}}\right) d\tilde{N}(s)\right. \\ & \left. - \int_0^t \left(\frac{\|q(s)\|^2}{2} + \lambda^{\mathbb{P}} - \lambda(s)\right) ds\right\}, \qquad 0 \le t \le T, \end{aligned} \tag{5.5}$$

with $\frac{\lambda(s)}{\lambda^{\mathbb{P}}} := \left(\frac{\lambda_1(s)}{\lambda_1^{\mathbb{P}}}, \ldots, \frac{\lambda_k(s)}{\lambda_k^{\mathbb{P}}}\right)^T$. In particular, $\mathbb{Q}$ is uniquely characterized by q and λ. The stochastic exponential on the right-hand side of (5.5) is also referred to as the Doléans-Dade exponential. By Girsanov's theorem, under $\mathbb{Q}$, $W_{\mathbb{Q}}(t) := W(t) - \int_0^t q(s)\,ds$ is a Brownian motion and the process $\tilde{N}(t)$ has jumps with intensity $\lambda(t)$. The probabilistic model $\mathbb{P}$ occurs when $q = 0$ and $\lambda = \lambda^{\mathbb{P}}$.

We then state the form of a penalty function induced by requiring a dynamic evaluation to be time-consistent (or, equivalently, by requiring recursiveness or Bellman's dynamic programming principle). The result is due to Tang and Wei [2012], who generalized a result of Delbaen and Gianin [2010] obtained in a Brownian setting to a setting with jumps.

Theorem 52 (Tang and Wei [2012]) *Suppose that for $0 \le t \le T$ we have $U(t,H) = \inf_{\{\mathbb{Q}\sim\mathbb{P} \text{ on } \mathcal{F}_t\}} \left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}[H] + c(t,\mathbb{Q})\right\}$. Then the following statements are equivalent:*

(i) U is time-consistent on square-integrable rewards.

(ii) U is recursive, i.e., U satisfies Bellman's principle: for every $0 \le t \le T$, $A \in \mathcal{F}_t$ and square-integrable H,

$$U(0, U(t,H)I_A) = U(0, HI_A).$$

(iii) There exists a function

$$\begin{array}{rccccccccc} r: & [0,T] & \times & \Omega & \times & \mathbb{R}^d & \times & \mathbb{R}^k & \to & \mathbb{R}\cup\{\infty\} \\ & (t, & & \omega, & & q, & & v) & \longmapsto & r(t,\omega,q,v), \end{array}$$

which is convex and lower semi-continuous in (q, v), such that

$$c(t, \mathbb{Q}) = \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[\int_t^T r(s, q(s), \lambda(s) - \lambda^{\mathbb{P}})ds\right], \qquad 0 \leq t \leq T. \tag{5.6}$$

Remark 53 *In the case of a coherent risk measure, (5.6) corresponds to the existence of a convex, closed, set-valued predictable mapping, say C, taking values in $\mathbb{R}^d \times \mathbb{R}^k$ such that $r(s, q, v) = I_{C_s}(q, v)$.*

Violation of time-consistency would lead to situations in which the decision-maker takes decisions that he knows he will regret in every future state of nature. We rule out such situations. Because in our continuous-time setting time-consistency is equivalent to a penalty function of the form (5.6), we henceforth assume:

(G1) $(c(t, \mathbb{Q}))_{0 \leq t \leq T}$ is of the form

$$c(t, \mathbb{Q}) = \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[\int_t^T r(s, q(s), \lambda(s) - \lambda^{\mathbb{P}})ds\right], \tag{5.7}$$

for a function $r : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}_0^+ \cup \{\infty\}$ that is lower semi-continuous and convex in (q, v) with $r(s, 0, 0) = 0$.

Remark 54 *We note that for numerical tractability of the optimal stopping problem, we have postulated in (G1) that r does not depend on ω.*

Remark 55 *Since by (G1) in particular $c(t) \geq 0$ and $c(t, \mathbb{P}) = 0$, we have $U(t, H) = H$ if H is $\mathcal{F}_t$-measurable. That is, if a reward is known, then there is no uncertainty, and therefore the evaluation returns the reward itself.*

We note that q may be viewed as an additional drift in the Brownian motion that the reference model $\mathbb{P}$ fails to detect, while $\lambda(s) - \lambda^{\mathbb{P}}$ is the deviation of the new jump intensity $\lambda(s)$ under $\mathbb{Q}$ from the intensity $\lambda^{\mathbb{P}}$ under $\mathbb{P}$. Since r is non-negative and $r(s, 0, 0) = 0$, r is minimal in $(0, 0)$ with $q = 0$ and $\lambda = \lambda^{\mathbb{P}}$. These values of q and λ render the probabilistic model $\mathbb{P}$ itself. Therefore, the reference model $\mathbb{P}$ is associated with the highest plausibility. (Note that, if we would not make the assumption that $r(s, 0, 0) = 0$, we could redefine the reference model $\mathbb{P}$ to correspond to a (q, λ) for which the minimum is attained.) The fact that $(q, \lambda - \lambda^{\mathbb{P}}) \mapsto r(t, q, \lambda - \lambda^{\mathbb{P}})$ is convex in $(q, \lambda - \lambda^{\mathbb{P}})$ (with minimum assumed to be in $(0, 0)$) explicates that penalty functions giving rise to time-consistent evaluations in our setting may be interpreted as penalty functions for which the divergence penalty function r is directly applied to the additional stochastic drift q affecting the Brownian motion and the deviation of the jump intensity $\lambda - \lambda^{\mathbb{P}}$, instead of to the composition of q and $\lambda - \lambda^{\mathbb{P}}$ appearing in the Radon-Nikodym derivative process (5.5).

We now illustrate the generality of (5.4) and (G1) with some examples of penalty functions satisfying our conditions:

Examples 56 (1) Kullback-Leibler divergence: A prototypical example of the penalty function in (5.4) is the Kullback-Leibler (ϕ-)divergence given by

$$c(t,\mathbb{Q}) = \alpha \mathrm{KL}_t(\mathbb{Q}|\mathbb{P}), \quad \alpha > 0, \text{ with} \mathrm{KL}_t(\mathbb{Q}|\mathbb{P}) = \begin{cases} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], & \text{if } \mathbb{Q} \in \mathcal{Q}; \\ \infty, \text{ otherwise}; \end{cases}$$

see Csiszár [1975] and Ben-Tal [1985]. The Kullback-Leibler divergence is also referred to as the relative entropy and measures the distance between the probabilistic models $\mathbb{Q}$ and $\mathbb{P}$; it is used e.g., by Hansen and Sargent [2001] and Hansen and Sargent [2007] to generate model robustness in macroeconomics. The interpretation is that the economic agent has a reference measure $\mathbb{P}$, but the measure $\mathbb{P}$ is merely an approximation to the probabilistic model rather than the true model. As such, the agent does not fully trust the measure $\mathbb{P}$ and considers many measures $\mathbb{Q}$, with esteemed plausibility decreasing proportionally to their distance from the approximation $\mathbb{P}$. The parameter α measures the degree of trust the decision-maker assigns to the reference model $\mathbb{P}$. The limiting case $\alpha \uparrow \infty$ ($\alpha \downarrow 0$) induces a maximal degree of trust (distrust). One may verify (see, for example, Proposition 9.10 in Cont and Tankov [2004]) that in our continuous-time setting, for every $\mathbb{Q}$ satisfying $c(\mathbb{Q}) < \infty$, $\alpha \mathrm{KL}_t(\mathbb{Q}|\mathbb{P})$ is of the form (5.7), where $r(s,q,v) = \frac{\alpha}{2}\|q\|^2 + \alpha(\lambda^{\mathbb{P}} + v)^T \log(\mathbf{1} + \frac{v}{\lambda^{\mathbb{P}}}) - \mathbf{1}^T v$ with $\mathbf{1} = (1,\dots,1)^T$, where q and λ correspond to the model $\mathbb{Q}$ characterized through (5.5), and where the logarithm should be taken componentwise.

(2) Worst case with discrete scenarios: The decision-maker considers a family of finitely many values $q_1(s),\dots,q_L(s)$ and $\lambda_1(s),\dots,\lambda_L(s)$ for the future drift, $q(s)$, and jump intensity, $\lambda(s)$, that characterize the model $\mathbb{Q}$ through (5.5), with $s > t$. Ex ante these L "scenarios" are equally plausible and the decision-maker adopts a worst case approach. Consider

$$M = \Big\{\mathbb{Q} \in \mathcal{Q} \Big| \quad \text{for Lebesgue-a.s. all } s: \\ (q(s), \lambda(s)) \in \{(q_i(s), \lambda_j(s)) | \, i,j \in \{1,\dots,L\}\}\Big\}.$$

This corresponds to a penalty function of the form (5.7), with

$$r(s,q,\lambda-\lambda^{\mathbb{P}}) = \begin{cases} 0, & \text{if} \quad (q,\lambda) \in \mathrm{conv}\Big(\{(q_i(s), \lambda_j(s))|i,j \in \{1,\dots,L\}\}\Big); \\ \infty, & \text{otherwise}; \end{cases}$$

where $\mathrm{conv}(\cdot)$ is given by its convex hull. (By redefining the reference measure, one may ensure (without loss of generality) that $0 \in \mathrm{conv}\Big(\{(q_i(s), \lambda_j(s))|i,j \in \{1,\dots,L\}\}\Big)$.

(3) Worst case with ball scenarios: The decision-maker considers alternative and equally plausible probabilistic models $\mathbb{Q}$ in a small ball around the reference model $\mathbb{P}$

and adopts a worst case approach:

$$M = \left\{ \mathbb{Q} \in \mathcal{Q} \middle| \left\| q\left(t\right) \right\| \leq \delta_1, \quad \left\| \lambda\left(t\right) \right\| \leq \delta_2, \text{ for Lebesgue-a.s. all } t \right\},$$

for $\delta_1, \delta_2 > 0$. This corresponds to a penalty function of the form (5.7), with

$$r(s, q, \lambda - \lambda^{\mathbb{P}}) = \begin{cases} 0, & \text{if} \quad \|q\| \leq \delta_1, \quad \left\| \lambda - \lambda^{\mathbb{P}} \right\| \leq \delta_2; \\ \infty, & \text{otherwise.} \end{cases}$$

For our next examples we will assume that the n-dimensional Markovian process $(X\left(t\right))_{0 \leq t \leq T}$ is either a geometric Brownian motion with jumps and drift, or a Brownian-Poisson process with drift. In the first case,

$$\frac{dX_i\left(t\right)}{X_i\left(t\right)} = \mu_i dt + \sigma_i dW\left(t\right) + J_i d\tilde{N}\left(t\right), \qquad i = 1, \ldots, n, \tag{5.8}$$

while in the second case

$$dX_i\left(t\right) = \mu_i dt + \sigma_i dW\left(t\right) + J_i d\tilde{N}\left(t\right), \qquad i = 1, \ldots, n, \tag{5.9}$$

for $\mu_i \in \mathbb{R}$, $\sigma_i \in \mathbb{R}^{1\times d}$, and $J_i \in \mathbb{R}^{1\times k}$. We set $\mu = (\mu_1, \ldots, \mu_n)^T \in \mathbb{R}^n$, $\sigma = (\sigma_1, \ldots, \sigma_n)^T \in \mathbb{R}^{n\times d}$ and $J = (J_1, \ldots, J_n)^T \in \mathbb{R}^{n\times k}$. In optimal entrance/exit decision problems, such as those provided in the Introduction, X often times satisfies either (5.8) or (5.9) (with or without jumps). In finance, μ_i is commonly referred to as the excess return and represents the compensation for bearing the risky asset i. Now let us continue with some examples of penalty functions that induce time-consistent evaluations, i.e., satisfy (G1), and may be considered in the general problem (5.4), assuming dynamics as in (5.8) or (5.9).

Examples 5 (Continued; with (5.8) or (5.9) valid)

(4) Worst case with mean (partially) known: The decision-maker is certain that the (instantaneous or logarithmic instantaneous) mean return μ lies between a known lower and upper bound, (μ^-) and (μ^+), respectively. As a special case, (μ^-) and (μ^+) coincide (mean fully known). By Girsanov's theorem, under $\mathbb{Q}$,

$$\mu^{\mathbb{Q}}\left(t\right) = \mu + \sigma q\left(t\right) + J\left(\lambda\left(t\right) - \lambda^{\mathbb{P}}\right).$$

The resulting models are considered equally plausible and the decision-maker adopts a worst case approach:

$$\begin{aligned} M &= \left\{ \mathbb{Q} \in \mathcal{Q} \middle| \mu^-\left(t\right) \leq \mu^{\mathbb{Q}}\left(t\right) \leq \mu^+\left(t\right), \quad \text{for Lebesgue-a.s. all } t \right\} \\ &= \left\{ \mathbb{Q} \in \mathcal{Q} \middle| \mu^-\left(t\right) - \mu \leq \sigma q\left(t\right) + J\left(\lambda\left(t\right) - \lambda^{\mathbb{P}}\right) \leq \mu^+\left(t\right) - \mu, \text{ for L.-a.s. all } t \right\}. \end{aligned}$$

However, in general M is non-compact, which often leads to a degenerate, non-semimartingale evaluation. We will therefore assume that the decision-maker only considers additional drifts and additional jump intensities between certain bounds, i.e., he only considers additional drifts q satisfying $B^- \leq q \leq B^+$ for certain vectors $B^+, B^- \in \mathbb{R}^n$ and jump intensities λ satisfying $d^- \leq \lambda - \lambda^{\mathbb{P}} \leq d^+$ for vectors $d^+, d^- > -\lambda^{\mathbb{P}}$, to ensure well-posedness. This corresponds to a penalty function of the form (5.7) with

$$r\left(s, q, \lambda - \lambda^{\mathbb{P}}\right) = \begin{cases} 0, & \text{if} \quad \mu^- - \mu \leq \sigma q + J\left(\lambda - \lambda^{\mathbb{P}}\right) \leq \mu^+ - \mu; \\ \infty, & \text{otherwise.} \end{cases}$$

(5) Pricing with Good-Deal Bounds: A fundamental approach to price financial derivatives that are liquidly traded on the financial market is by replicating the derivatives using other (base) assets and applying no-arbitrage arguments. However, if the financial market is incomplete, a full-blown replication is infeasible, and no-arbitrage arguments only yield price bounds. In general, these price bounds are typically too wide to be practically useful. One approach to narrowing these bounds is provided by the good-deal pricing approach introduced by Cochrane and Saá-Requejo [2000]. Under this approach, only pricing kernels that are sufficiently "close" to the physical measure are considered. Here, "close" means that only pricing kernels with a variance below a certain bound are considered. By duality results derived in a celebrated paper by Hansen and Jagannathan [1991], this corresponds to ruling out portfolios with a too high Sharpe ratio. The intuition is that portfolios with a very high Sharpe ratio, although strictly speaking not providing arbitrage opportunities, are "too good to be true" and will be eliminated in a competitive market. In a continuous-time setting, such as ours, the bound for the variance of the pricing kernel is equal to the highest (local) Sharpe ratio in the market, say Λ. In this case, the good-deal bound evaluation $U(t, H(\tau))$ is given by

$$U(t, H(\tau)) = \inf_{(q,\lambda) \in C} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[H(\tau)\right],$$

with $C = (C_t)_{0 \leq t \leq T}$ given by (see Björk and Slinko [2006])

$$C_t = \Big\{ \left(q, \lambda - \lambda^{\mathbb{P}}\right) \in \mathbb{R}^d \times \mathbb{R}^k \ \Big| \mu + \sigma q + J\left(\lambda - \lambda^{\mathbb{P}}\right) = 0 \\ \text{and } \|q\|^2 + \textstyle\sum_{i=1}^k \frac{(\lambda_i - \lambda_i^{\mathbb{P}})^2}{\lambda_i^{\mathbb{P}}} \leq \Lambda^2 \Big\}.$$

This corresponds to a penalty function of the form (5.7) with

$$r\left(s, q, \lambda - \lambda^{\mathbb{P}}\right) = \begin{cases} 0, & \text{if} \quad (q, \lambda) \in C; \\ \infty, & \text{otherwise.} \end{cases}$$

For numerical tractability in what follows, we need the following additional assumption:

(G2) We can simulate i.i.d. copies of $(X(t))_{0 \leq t \leq T}$.

Assumption (G2) is satisfied in particular if X follows a linear SDE, which holds e.g., in the case of a Brownian motion with drift, a Poisson process with drift, an Ornstein-Uhlenbeck process, or a geometric Brownian motion with drift (with or without Poisson type jumps). But note there are by now also very general results available on exact sampling of more general diffusions and jump-diffusions; see, e.g., Beskos and Roberts [2005], Broadie and Kaya [2006], Chen and Huang [2014], or Giesecke and Smelov [2014(@].

In principle, we would only need assumptions (G1)-(G2). However, if the sub-level sets of the penalty function are non-compact (meaning that models that are "far away" from the reference model may still yield high plausibility), then the associated optimal stopping problem (5.4) would be ill-posed. To verify, consider, for example, the case that $c = 0$ so that $U(0, H(\tau)) = \inf_\omega H(\tau(\omega), \omega)$, which leads to a degenerate (and non-semimartingale) evaluation. Therefore, we will assume additionally to (G1)-(G2) that:

(G3) The domain of r is included in a compact set: for every s,

$$\left\{(q, \lambda) \in \mathbb{R}^d \times \mathbb{R}^k \middle| \; r\left(s, q, \lambda - \lambda^{\mathbb{P}}\right) < \infty\right\} \subset C_s,$$

for a compact set $C = (C_s)_{0 \leq s \leq T} \subset [0, T] \times \mathbb{R}^d \times \mathbb{R}^k_+$.

Loosely speaking, condition (G3) states that, if the additional drift q or jump intensity $\lambda - \lambda^{\mathbb{P}}$ that the model $\mathbb{Q}$ adds to the Brownian motion or the Poisson process when compared to $\mathbb{P}$ is "too large", then the model $\mathbb{Q}$ should not be considered. Condition (G3) may be generalized substantially. In fact, it would be sufficient for our purposes to impose a condition on the penalty function that guarantees that the sub-level sets are (weakly) compact. However, in order to keep the exposition simple, we will impose the somewhat stronger condition (G3).

5.2. Duality Theory

5.2.1. Duality Theory of the First Kind

Reconsider the optimal stopping problem (5.4). We show in the Section 5.5 that there exists an optimal stopping family $(\tau^*(t_i))_{t_i \in \{0=t_0, t_1, \ldots, t_L = T\}}$ satisfying

$$V^*(t_i) = \sup_{\tau \in \mathcal{T}_i} U(t_i, H(\tau)) = U(t_i, H(\tau^*(t_i))), \quad t_i \in \{0, \ldots, T\}. \tag{5.1}$$

Furthermore, we show that Bellman's principle

$$V^*(t_i) = \max\left(\Pi(t_i, X(t_i)) + U^h(t_i), U(t_i, V^*(t_{i+1}))\right), \quad t_i \in \{0, \ldots, t_{L-1}\}, \tag{5.2}$$

holds, with $U^h(t_i)$ defined as

$$\begin{aligned}U^h(t_i) &:= U\left(t_i, \sum_{j=i}^{L} h(t_j, X(t_j))\right)\\ &= \inf_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_{t_i}}\left[\sum_{j=i}^{L} h(t_j, X(t_j)) + \int_{t_i}^{T} r\left(s, q(s), \lambda(s) - \lambda^{\mathbb{P}}\right) ds\right]\right\};\end{aligned} \tag{5.3}$$

see the Section 5.5 for technical details. Recall that in the absence of model uncertainty, $U(t_i, H)$ reduces simply to an ordinary conditional expectation (corresponding to the case in which $c(t_i, \mathbb{Q}) = \infty$ for $\mathbb{Q} \neq \mathbb{P}$ and $c(t_i, \mathbb{P}) = 0$ in (5.3)).

To compute the solution V^* — referred to as the (generalized) Snell envelope — to the optimal stopping problem (5.4), we will rely on the Doob decomposition of V^* into a martingale and a predictable process. However, to do so, we first need to generalize the notion of a (standard) martingale (with respect to an ordinary conditional expectation) to martingales with respect to classes of functionals: We will say that M is a U-martingale if $M(s) = U(s, M(t))$, $s, t \in \{0 = t_0, t_1, \ldots, t_L = T\}$ and $s \le t$. By time-consistency, this is equivalent to $M(s) = U(s, M(T))$ for any s. The class of U-martingales M with $M(0) = 0$ is denoted by $\mathcal{M}_0^U$. Define, for $i = 0, \ldots, L$,

$$A^{*g}(t_i) := \sum_{j=1}^{i}(U(t_{j-1}, V^*(t_j)) - V^*(t_{j-1})), \quad M^{*g}(t_i) := \sum_{j=1}^{i}(V^*(t_j) - U(t_{j-1}, V^*(t_j))). \tag{5.4}$$

One may verify that M^{*g} is a U-martingale, A^{*g} is non-decreasing and predictable, $M^{*g}(0) = A^{*g}(0) = 0$, and that

$$V^*(t_i) = V^*(0) + M^{*g}(t_i) + A^{*g}(t_i), \qquad i = 0, \ldots, L, \tag{5.5}$$

provides a U-Doob decomposition of $V^* = (V^*(t_i))_{t_i \in \{0=t_0,\ldots,t_L=T\}}$.

To construct genuine upper bounds to the optimal solution to the stopping problem (5.4), which will converge asymptotically to the true value, our method will exploit an additive dual representation of the optimal stopping problem (5.4), by expanding the well-known dual representation for the classical setting, in which U is just the ordinary conditional expectation (Rogers [2002] and Haugh and Kogan [2004]). This generalized additive dual representation, the proof of which uses results obtained by Krätschmer and Schoenmakers [2010] in a discrete-time setting with $h = 0$, reads as follows:

Proposition 57 *Let $M^{*g} \in \mathcal{M}_0^U$ be the (unique) U-martingale in the U-Doob decomposition (5.5). Then the optimal stopping problem (5.4) has a dual representation*

$$\begin{aligned}V^*(t_i) &= \inf_{M\in\mathcal{M}_0^U} U\left(t_i, \max_{t_j\in\{t_i,\ldots,T\}}\left(\Pi(t_j, X(t_j)) + U^h(t_j) + M(T) - M(t_j)\right)\right)\\ &= U\left(t_i, \max_{t_j\in\{t_i,\ldots,T\}}\left(\Pi(t_j, X(t_j)) + U^h(t_j) + M^{*g}(T) - M^{*g}(t_j)\right)\right),\end{aligned} \tag{5.6}$$

where $t_i \in \{0 = t_0, ..., t_L = T\}$.

Remark 58 *In the absence of model uncertainty, so that* U *is a regular conditional expectation,* $\mathcal{M}_0^U = \mathcal{M}_0$ *is the class of martingales in the usual sense. This result was shown in Rogers [2002] and also used in Andersen and Broadie [2004] and Haugh and Kogan [2004]. In this case, interestingly, also*

$$V^*(t_i) = \inf_{M \in \mathcal{M}_0} U\left(t_i, \max_{t_j \in \{t_i, \dots, T\}} \left(\Pi(t_j, X(t_j)) + U^h(t_j) + M(t_i) - M(t_j)\right)\right), \quad (5.7)$$

$t_i \in \{0 = t_0, ..., t_L = T\}$, *is true. So, for regular conditional expectations, in fact two dual representations hold, namely (5.6) and (5.7). However, (5.7) breaks down in general if* U *is not a conditional expectation, and only (5.6) is preserved.*

5.2.2. Duality Theory of the Second Kind

Next, we describe the second kind of duality theory on which our method is based. For $0 \leq t \leq T$, $z \in \mathbb{R}^{1 \times d}$ and $\tilde{z} \in \mathbb{R}^{1 \times k}$, given a function r specifying the penalty function c through (5.7), we define a function g by Fenchel's duality as follows:

$$g(t, z, \tilde{z}) := \inf_{(q, \lambda - \lambda^{\mathbb{P}}) \in C_t} \left\{ zq + \tilde{z}\left(\lambda - \lambda^{\mathbb{P}}\right) + r\left(t, q, \lambda - \lambda^{\mathbb{P}}\right)\right\}, \quad (5.8)$$

with C_t induced by assumption (G3). Note that by assumption (G3), g thus defined is Lipschitz continuous. Note furthermore that (G3) is satisfied in all our Examples 56 above, except for the Kullback-Leibler divergence. In this case, however, we will restrict our analysis to terminal conditions that are Lipschitz continuous in the Brownian motion and the Poisson process, so that the domains of z and $\tilde{z}$ are bounded, and g may be considered to be Lipschitz continuous as well. Furthermore, suppose that, for every exercise date t_j, $j = 0, \dots, L$, we have a fine time grid $\pi_j = \{s_{j0} = t_j < s_{j1} < \dots < s_{jP} = t_{j+1}\}$. Denote the corresponding overall time grid by $\pi = \{s_{00}, s_{01}, \dots, s_{LP}\}$. The following theorem provides a way to practically compute M^{*g} in (5.4) by connecting it to specific semi-martingale dynamics that can be dealt with numerically efficiently.

Theorem 59 *(a) There exist unique square integrable predictable* $(Z^h, \tilde{Z}^h)$ *such that*

$$dU^h(t) = -g\left(t, Z^h(t), \tilde{Z}^h(t)\right) dt + Z^h(t)\, dW(t) + \tilde{Z}^h(t)\, d\tilde{N}(t), \quad (5.9)$$

for $0 < t_j \leq t_{j+1}$ *and* $U^h(t_j) = U^h(t_j+) + h(t_j, X(t_j))$, *for each* $j \in \{0, \dots, L-1\}$. *Furthermore, there exist unique square-integrable predictable* $(Z^*, \tilde{Z}^*)$ *such that*

$$dU(t, V^*(t_{j+1})) = -g\left(t, Z^*(t), \tilde{Z}^*(t)\right) dt + Z^*(t)\, dW(t) + \tilde{Z}^*(t)\, d\tilde{N}(t), \quad (5.10)$$

for $0 \leq t_j \leq t_{j+1}$, $j \in \{0, ..., L-1\}$.

(b) For $0 \leq t \leq T$, $(Z^*, \tilde{Z}^*)$ from part (a) satisfy

$$\begin{aligned} M^{*g}(t) &= U(t, M^{*g}(T)) = -\int_0^t g\left(t, Z^*(t), \tilde{Z}^*(t)\right) dt \\ &\quad + \int_0^t Z^*(t)\, dW(t) + \int_0^t \tilde{Z}^*(t)\, d\tilde{N}(t)\,. \end{aligned} \tag{5.11}$$

Remark 60 *Note that we have terminal conditions given by $U^h(T) = h(T, X(T))$ and $U(t_{j+1}, V^*(t_{j+1})) = V^*(t_{j+1})$, for $j = 0, \ldots, L-1$, in (5.9) and (5.10) by Remark 55 and (5.3). Hence, given $U^h(t_{j+1})$ and $V^*(t_{j+1})$, we may compute $U^h(t_j)$ and $U(t_j, V^*(t_{j+1}))$ through the relationships given in Theorem 59(a); $V^*(t_j)$ can then be obtained by Bellman's principle (5.2).*

Remark 61 *As $U(t_{j+1}, V^*(t_{j+1})) = V^*(t_{j+1})$, we can write, by Theorem 59(a), for $t_j \leq t \leq t_{j+1}$,*

$$\begin{aligned} U(t, V^*(t_{j+1})) &= V^*(t_{j+1}) + \int_t^{t_{j+1}} g\left(s, Z^*(s), \tilde{Z}^*(s)\right) ds \\ &\quad - \int_t^{t_{j+1}} Z^*(s)\, dW(s) - \int_t^{t_{j+1}} \tilde{Z}^*(s)\, d\tilde{N}(s)\,. \end{aligned} \tag{5.12}$$

Similarly, it follows that, for $t_j < t \leq t_{j+1}$,

$$\begin{aligned} U^h(t) &= U^h(t_{j+1}) + \int_t^{t_{j+1}} g\left(s, Z^h(s), \tilde{Z}^h(s)\right) ds \\ &\quad - \int_t^{t_{j+1}} Z^h(s)\, dW(s) - \int_t^{t_{j+1}} \tilde{Z}^h(s)\, d\tilde{N}(s)\,. \end{aligned} \tag{5.13}$$

Remark 62 *Note that if $g = 0$ would hold in (5.10), then the increments of the evaluation U were increments of a (standard) martingale. In that case, $U(t, H)$ would simply be a (standard) martingale, and, because $U(T, H) = H$, correspond to the (regular) conditional expectation $U(t, H) = \mathbb{E}^{\mathcal{F}_t}[H]$. However, our decision-maker is ambiguity averse and considers alternative probabilistic models with potentially different degrees of esteemed plausibility. This leads to $g \leq 0$, which by (5.12)-(5.13) decreases the evaluation. Note furthermore that the couple Z^* and $\tilde{Z}^*$ may be viewed as a measurement of the degree of "variability" underlying the evaluation — in the same way as the volatility in standard asset pricing models in finance — due to the Brownian motion and the jump component, respectively: The larger $\|Z^*\|$, $\left(\left\|\tilde{Z}^*\right\|\right)$, the more variability comes from the local Gaussian part (the jump component) of the model. Because $g(t, \cdot) \leq 0$ is concave* *in $(z,\tilde{z})$, with maximum in $(0,0)$, greater variability will lead to a larger "penalty" term. In (5.9), the reward h has an opposite sign compared to g. Hence, in U_t^h, we encounter a trade-off between obtaining a continuously paid future reward and ambiguity aversion.*

Remark 63 *Furthermore, the Theorem shows that an optimal entrance problems under ambiguity corresponds to an American option problem with payoffs given by the fixed costs $\Pi(t,X(t))$ and with an additional drift $h-g$ which can be interpreted as the reward (arising from the decision to enter the market) minus a penalty respectively (arising from the ambiguity aversion).*

Equations (5.9)–(5.10) are also referred to as backward stochastic differential equations (BSDE)[1] and their solution is often referred to as a (conditional) g-expectation. A g-expectation inherits many properties from a regular (conditional) expectation, such as monotonicity, translation invariance, and the tower property, but not linearity; for further details, see, for instance, the survey of Peng [1997].

To conclude the exposition of the duality theory of the second kind, let us, for illustration purposes, employ the penalty functions of Examples 56 and compute the corresponding g's using (5.8). These g functions will later be used in numerical illustrations.

Examples 64 (1) Kullback-Leibler divergence: As Fenchel's dual conjugate of $\alpha((1+x)\log(1+x)-x)$ is given by $\Phi(y):=\alpha(e^{y/\alpha}-\frac{y}{\alpha}-1)$, we have with (5.8)

$$\begin{aligned} g(t,z,\tilde{z}) &= \inf_{(q,\lambda)\in C_t}\left\{zq+\tilde{z}\lambda+r\left(t,q,\lambda-\lambda^{\mathbb{P}}\right)\right\} \\ &= \inf_{(q,\lambda-\lambda^{\mathbb{P}})\in C_t}\left\{zq+\frac{\alpha}{2}|q|^2\right\}+\lambda^{\mathbb{P}}\inf_{(q,\lambda-\lambda^{\mathbb{P}})\in C_t}\left\{-(-\tilde{z})\frac{\lambda-\lambda^{\mathbb{P}}}{\lambda^{\mathbb{P}}}\right. \\ &\quad \left.+\left[\alpha\left(1+\frac{\lambda-\lambda^{\mathbb{P}}}{\lambda^{\mathbb{P}}}\right)^T\log\left(\mathbf{1}+\frac{\lambda-\lambda^{\mathbb{P}}}{\lambda^{\mathbb{P}}}\right)-\mathbf{1}^T\frac{\lambda-\lambda^{\mathbb{P}}}{\lambda^{\mathbb{P}}}\right]\right\} \\ &= -\frac{\|z\|^2}{2\alpha}-\alpha\sum_{i=1}^k\lambda_i^{\mathbb{P}}\left(e^{-\tilde{z}_i/\alpha}+\frac{\tilde{z}_i}{\alpha}-1\right), \end{aligned}$$

where the infimum is attained at $q=-\frac{z}{\alpha}$ and the second part follows with results of convex optimization. For a functional $f:X\to\mathbb{R}$ it holds $-f^*(x^*)=\inf_{x\in X}\{f(x)-\langle x^*,x\rangle\}$ with $f^*:X^*\to\mathbb{R}$ being the dual conjugate.

$$g(t,z,\tilde{z})=-\frac{\|z\|^2}{2\alpha}-\alpha\sum_{i=1}^k\lambda_i\left(e^{-\tilde{z}_i/\alpha}+\frac{\tilde{z}_i}{\alpha}-1\right).$$

[1] Formally, given a terminal payoff $H\in L^2$ and a function $g:[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\to\mathbb{R}$, the solution to the corresponding BSDE is a triple of square-integrable and suitably measurable processes $(Y,Z,\tilde{Z})$ satisfying

$$dY_t=-g(t,Z_t,\tilde{Z}_t)dt+Z_tdW_t+\tilde{Z}_td\tilde{N}_t,\quad\text{and}\quad Y_T=H.$$

(2) Worst case with discrete scenarios: It is straightforward to verify that in the case of a worst case scenario evaluation we obtain

$$g(t,z,\tilde{z}) = \min_{i=1,\dots,L} zq_i(t) + \min_{i=1,\dots,L} \tilde{z}\left(\lambda_i - \lambda^{\mathbb{P}}\right).$$

(3) Worst case with ball scenarios: Suppose without loss of generality that $||\lambda^{\mathbb{P}}|| \geq \delta_2$. Then

$$g(t,z,\tilde{z}) = -\delta_1 \|z\| - \delta_2 \|\tilde{z}\|.$$

(4) Worst case with mean (partially) known and (5.8) or (5.9): From (5.8),

$$g(t,z,\tilde{z}) = \inf_{(q,\lambda-\lambda^{\mathbb{P}})\in C_t} \left\{zq + \tilde{z}\left(\lambda - \lambda^{\mathbb{P}}\right)\right\}, \tag{5.14}$$

with

$$C_t = \Big\{ \left(q, \lambda - \lambda^{\mathbb{P}}\right) \in \mathbb{R}^d \times \mathbb{R}^k \Big| \mu^- - \mu \leq \sigma q + J\left(\lambda - \lambda^{\mathbb{P}}\right) \leq \mu^+ - \mu,$$
$$B^- \leq q \leq B^+, \quad d^- \leq \lambda - \lambda^{\mathbb{P}} \leq d^+ \Big\}.$$

In general, g cannot be simplified further, although it can in specific cases, such as $(\mu^-) = (\mu^+)$ (mean fully known). However, in view of (5.14), for fixed $(t,z,\tilde{z})$, g can be obtained as the solution to a linear programming problem.

(5) Good-Deal Bounds and (5.8) or (5.9): Let $b = -\mu$ and let $A = (\sigma, J)$ be a matrix mapping from $\mathbb{R}^d \times \mathbb{R}^k$ to $\mathbb{R}^n$. Note that there exists a unique decomposition

$$(z,\tilde{z}) = P(z,\tilde{z}) + (z,\tilde{z})^{\perp},$$

where $P(z,\tilde{z})$ denotes the projection of $(z,\tilde{z})$ on the image of A^T. Define $\left\langle (z,\tilde{z}), \left(q, \lambda - \lambda^{\mathbb{P}}\right)\right\rangle := qz + \tilde{z}\left(\lambda - \lambda^{\mathbb{P}}\right)$. Furthermore, for $q \in \mathbb{R}^d$ and $v \in \mathbb{R}^k$, define $|\langle q, v\rangle|_* := \sqrt{\|q\|^2 + \left\|\frac{v}{\lambda^{\mathbb{P}}}\right\|^2}$, where the division is defined componentwise. First of all, it is not hard to see that the dual of $(q,v) \to |\langle q,v\rangle|_*$ is given by $|\langle z, \tilde{z}\rangle|_{**} := \sqrt{\|z\|^2 + \|\tilde{z}\cdot\lambda^{\mathbb{P}}\|^2}$, where "$\cdot$" denotes componentwise multiplication. Then,

$$g(t,z,\tilde{z}) = \inf_{(q,\lambda-\lambda^{\mathbb{P}})\in C} \left\langle (z,\tilde{z}), \left(q, \lambda - \lambda^{\mathbb{P}}\right)\right\rangle,$$

with C given by

$$C = \left\{ \left(q, \lambda - \lambda^{\mathbb{P}}\right) | A\left(q, \lambda - \lambda^{\mathbb{P}}\right)^T = b \quad \text{and} \quad \left|\left\langle q, \lambda - \lambda^{\mathbb{P}}\right\rangle\right|_* \leq \Lambda \right\}.$$

(Note that the case of no-arbitrage pricing corresponds to $\Lambda = \infty$.) If the set C is non-empty, this optimization problem has an explicit solution: Let $P_W(0)$ be the projection of 0 onto the set $W := \{x | Ax = b\}$ in the $|\cdot|_*$ norm. Using Lagrangian duality techniques, we have that

$$
\begin{aligned}
g(t,z,\tilde{z}) &= \inf_{(q,\lambda-\lambda^{\mathbb{P}})\in C} \left\langle (z,\tilde{z}), \left(q,\lambda-\lambda^{\mathbb{P}}\right)\right\rangle \\
&= \inf_{(q,\lambda-\lambda^{\mathbb{P}})\in C} \left\langle (z,\tilde{z}), \left(q,\lambda-\lambda^{\mathbb{P}}\right) - P_W(0)\right\rangle + \langle (z,\tilde{z}), P_W(0)\rangle \\
&= \inf_{\substack{\left(q,\lambda-\lambda^{\mathbb{P}}\right) - P_W(0) \in \mathrm{Kern}(A), \\ \left|\left(q,\lambda-\lambda^{\mathbb{P}}\right)\right|_* \le \Lambda}} \left\langle P(z,\tilde{z}) + (z,\tilde{z})^{\perp}, \left(q,\lambda-\lambda^{\mathbb{P}}\right) - P_W(0)\right\rangle \\
&\quad + \langle (z,\tilde{z}), P_W(0)\rangle \\
&= \inf_{|(q,\lambda-\lambda^{\mathbb{P}})-P_W(0)|_* \le \Lambda - |P_W(0)|_*} \left\langle (z,\tilde{z})^{\perp}, \left(q,\lambda-\lambda^{\mathbb{P}}\right) - P_W(0)\right\rangle \\
&\quad + \langle (z,\tilde{z}), P_W(0)\rangle \\
&= -(\Lambda - |P_W(0)|_*)\sqrt{\|z\|^2 + \|\tilde{z}\cdot\lambda^{\mathbb{P}}\|^2} + \langle (z,\tilde{z}), P_W(0)\rangle \\
&= -(\Lambda - |P_W(0)|_*)\sqrt{||z||^2 + \Big|\sum_{i=1}^{k} \tilde{z}_i \lambda_i^{\mathbb{P}}\Big|^2} + \langle (z,\tilde{z}), P_W(0)\rangle\,,
\end{aligned}
$$

where we used in the fourth equation the definition of $P_W(0)$. Moreover, we applied that $(z,\tilde{z})^{\perp}$ lies in the dual space of $\mathrm{Kern}(A)$ and that therefore the constraint $\left(q,\lambda-\lambda^{\mathbb{P}}\right) - P_W(0) \in \mathrm{Kern}(A)$ may be omitted. This concludes our examples.

5.3. The Algorithm

5.3.1. General Outline

Our method is composed of four steps. Theorem 59 ("duality theory of the second kind") jointly with Bellman's principle (5.2) will serve as a *first* stepping stone for our approach, by providing a practical way to find U-martingales to be employed in the dual representation (5.6), which is our *second* stepping stone ("duality theory of the first kind"). In particular, Theorem 59(a) yields that, to construct the U-martingale M^{*g} in the U-Doob decomposition (5.5) of the (generalized) Snell envelope V^* solving our optimal stopping problem, we only have to find $(Z^*, \tilde{Z}^*)$ from the martingale part for every $(V^*(s))_{t_j<s\le t_{j+1}}$. And this can be achieved either by solving a PDE (or PIDE in the presence of jumps) or by least squares Monte Carlo regression and backward stochastic calculus. We will adopt the latter approach. Further we construct the continuation value which will be used as an exercise criterion in step four. It will provide an approximated upper bound on the solution V^* to the optimal stopping problem, in view of the dual representation (5.6) in Proposition 57. While this bound will be seen to converge to the true optimal solution asymptotically and is an approximated upper bound at the pre-limiting level, it is not a genuine upper bound estimate to the true optimal solution as it is not "biased high", that is, biased above the Snell envelope V^*. This means that on average this upper bound may not provide enough protection. Our *third* stepping stone, then, is the introduction of backward-forward simulation in the context of BSDEs to

obtain a genuine (biased high) upper bound on the solution V^* to our stopping problem (see Step (3.) below). Finally we calculate a lower bound using the concept of BSDEs with random terminal time.

Therefore, we will:

Step (1.) Exploiting duality theory of the second kind:

Step (1.a.) Compute an approximation to $\left(U^h(t_j)\right)_{t_j \in \{0,\ldots,T\}}$ in (5.3) through backward recursion, using (5.9) and $U^h(T) = 0$. This involves least squares Monte Carlo regression.

Step (1.b.) Set $V^*(T) = H(T) = \Pi(T, X(T))$ and after that do a backward recursion over t_j: Then given $V^*(t_{j+1})$, compute $\left(Z^*(s), \tilde{Z}^*(s)\right)_{t_j \leq s \leq t_{j+1}}$ and $U(s, V^*(t_{j+1}))_{t_j < s \leq t_{j+1}}$ through (5.10). This step involves the application of least squares Monte Carlo regression. We can then set $V^*(t_j) = \max\left(\Pi(t_j, X(t_j)) + U^h(t_j), U(t_j, V^*(t_{j+1}))\right)$, by (5.2). If (and as long as) $t_j > 0$, set $j = j - 1$, and repeat the same computation. Otherwise, go to Step (1.c.) below.

Step (1.c.) Compute the continuation value defined by

$$U^c(t_j) = U(t_j, V^*(t_{j+1})).$$

Step (1.d.) Given the whole path of $\left(Z^*(s), \tilde{Z}^*(s)\right)_{0 \leq s \leq T}$, compute an approximation to $(M^{*g}(t_j))_{t_j \in \{t_1,\ldots,T\}}$ through (5.11).

Step (2.) Exploiting duality theory of the first kind, obtain an approximated upper bound to $V^*(0)$ through (5.6). This involves least squares Monte Carlo regression.

Step (3.) Introducing backward-forward simulation:

Step (3.a.) Compute a genuine (biased high) upper bound to $\left(U^h(t_j)\right)_{t_j \in \{0,\ldots,t_{L-1}\}}$ by using the least squares Monte Carlo results obtained under Step (1.a.) as input in Monte Carlo forward simulations.

Step (3.b.) Compute a genuine (biased high) upper bound to the Snell envelope $V^*(0)$ by using the least squares Monte Carlo results obtained under Steps (1.) and (2.) as input in Monte Carlo forward simulations.

Before we proceed to describe our algorithm (in particular, Steps (1.)–(3.) above) in more detail, we underline the following: Since our optimal stopping problem is Markovian, there exists a function $v^* : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that $V^*(t) = v^*(t, X(t))$. In particular, $V^*(0) = v^*(0, X(0))$. Our method will ultimately provide an approximation to the function v using Monte Carlo simulation techniques that are standard in e.g., the (no-ambiguity) American option literature. This entails that, for a finite number of Monte Carlo simulations, our approximation will inherently be random, as it depends on the stochastic nature of simulations. Our method, then, will be proven (see Theorem 67 below for the formal results) to have the following two appealing properties:

(i) Our approximation converges to its true value as the mesh size of the time grid tends to zero and the number of Monte Carlo simulations and basis functions tends to infinity.

(ii) For every finite time grid and finite number of Monte Carlo simulations and basis functions, our approximation provides a genuine (biased high) upper bound to its true value.

Our numerical examples show that, already after a limited number of simulations, our method yields rather close estimates in realistic settings. Moreover, by property (ii) above, for a finite time grid and a finite number of Monte Carlo simulations, our method will also provide a safety buffer, i.e., a maximal amount the decision-maker (firm or buyer) should be willing to pay or reserve for the action or undertaking.

5.3.2. Step-Wise Description

Step (1.a.): Construct an Approximation to U^h

Since the approximation scheme adopted in Step (1.a.) will also be used in the steps that follow, it will be useful to use slightly more general notation. We start with a function $w : \mathbb{R}^n \to \mathbb{R}$ (such that $w(X(T))$ is square-integrable) and the function $g(t, z, \tilde{z})$. Define $\Delta_{jp} := s_{j(p+1)} - s_{jp}$, $\Delta W_{jp} := W\left(s_{j(p+1)}\right) - W(s_{jp})$, $\Delta\tilde{N}_{jp} := \tilde{N}\left(s_{j(p+1)}\right) - \tilde{N}(s_{jp})$, and $|\pi| := \max_{j,p} \Delta_{jp}$, $j = 0, \ldots, L$, $p = 0, \ldots, P$. We will approximate U^h in (5.3) with a process Y^π. We initialize $Y^\pi(T) = y^\pi(X(T)) = w(X(T))$ where (here in Step (1.a.)) $w(X(T)) = h(T, X(T))$. We then do a backward recursion over the s_{jp}. Suppose we have an approximation $Y^\pi\left(s_{j(p+1)}\right)$ and we want to compute $Y^\pi(s_{jp})$. Theorem 59 then yields

$$Y^\pi(s_{jp}) \approx Y^\pi\left(s_{j(p+1)}\right) + g(s_{jp}, Z^\pi(s_{jp}), \tilde{Z}^\pi(s_{jp}))\Delta_{jp} - Z^\pi(s_{jp})\Delta W_{jp} - \tilde{Z}^\pi(s_{jp})\Delta\tilde{N}_{jp}$$

for all j, p, see (5.9). Taking conditional expectations,

$$Y^\pi(s_{jp}) \approx \mathbb{E}_{jp}\left[Y^\pi\left(s_{j(p+1)}\right)\right] + g(s_{jp}, Z^\pi(s_{jp}), \tilde{Z}^\pi(s_{jp}))\Delta_{jp}, \tag{5.1}$$

with $\mathbb{E}_{jp}[\cdot] = \mathbb{E}[\cdot|X(s_{jp})]$. We take

$$\left(Z^\pi(s_{jp}), \tilde{Z}^\pi(s_{jp})\right) = \underset{(Z,\tilde{Z})\in L^2_{d+k}(\sigma(X(s_{jp})))}{\arg\min} \mathbb{E}\left[\left(Y^\pi\left(s_{j(p+1)}\right) - Z\Delta W_{jp} - \tilde{Z}\Delta\tilde{N}_{jp}\right)^2\right].$$

Suppose that we have basis functions $(m_k(s_{jp}, X(s_{jp})))_{k\in\mathbb{N}}$, $(\psi_k(s_{jp}, X(s_{jp})))_{k\in\mathbb{N}}$ and $(\tilde{\psi}_k(s_{jp}, X(s_{jp})))_{k\in\mathbb{N}}$ for all j, p, spanning the space $L^2(\sigma(X(s_{jp})))$, respectively. Since we can computationally deal only with finitely many basis functions let us fix an $M \in \mathbb{N}$.

We write

$$m^M(s_{jp}, X(s_{jp})) = (m_1(s_{jp}, X(s_{jp})), \dots, m_M(s_{jp}, X(s_{jp})))^T,$$

and define ψ^M and $\tilde{\psi}^M$ similarly. Furthermore, define by $P^{\pi,M}_{s_{jp}}\left(Y^{\pi,M}\left(s_{j(p+1)}\right)\right) := \alpha^{\pi,M}_{s_{jp}} m^M(s_{jp}, X(s_{jp}))$, and

$$\begin{aligned} Z^{\pi,M}(s_{jp})\left(Y^{\pi,M}\left(s_{j(p+1)}\right)\right) &:= \gamma^{\pi,M}_{s_{jp}} \psi^M(s_{jp}, X(s_{jp}))\Delta W_{jp}, \\ \tilde{Z}^{\pi,M}(s_{jp})\left(Y^{\pi,M}\left(s_{j(p+1)}\right)\right) &:= \tilde{\gamma}^{\pi,M}_{s_{jp}} \tilde{\psi}^M(s_{jp}, X(s_{jp}))\Delta \tilde{N}_{jp}, \end{aligned}$$

the orthogonal projections on the space spanned by the basis functions $m^M(s_{jp}, X(s_{jp}))$, $\psi^M(s_{jp}, X(s_{jp}))\Delta W_{jp}$ and $\tilde{\psi}^M(s_{jp}, X(s_{jp}))\Delta \tilde{N}_{jp}$, respectively. (Here and in the remainder of this section, we understand vector multiplication as dot (scalar) product.) Note that

$$\begin{aligned} \alpha^{\pi,M}_{s_{jp}} &= \left(A^{\pi,M}_{s_{jp}}\right)^{-1} \mathbb{E}\left[Y^{\pi,M}\left(s_{j(p+1)}\right) m^M(s_{jp}, X(s_{jp}))\right], \\ \gamma^{\pi,M}_{s_{jp}} &= \left(\bar{A}^{\pi,M}_{s_{jp}}\right)^{-1} \mathbb{E}\left[Y^{\pi,M}\left(s_{j(p+1)}\right) \psi^M(s_{jp}, X(s_{jp}))\Delta W_{jp}\right], \qquad (5.2) \\ \tilde{\gamma}^{\pi,M}_{s_{jp}} &= \left(\tilde{A}^{\pi,M}_{s_{jp}}\right)^{-1} \mathbb{E}\left[Y^{\pi,M}\left(s_{j(p+1)}\right) \tilde{\psi}^M(s_{jp}, X(s_{jp}))\Delta \tilde{N}_{jp}\right], \qquad (5.3) \end{aligned}$$

with coefficients given by

$$\begin{aligned} \left(A^{\pi,M}_{s_{jp}}\right)_{1\le k,l\le M} &= \mathbb{E}\left[m^M_k(s_{jp}, X(s_{jp})) m^M_l(s_{jp}, X(s_{jp}))\right], \\ \left(\bar{A}^{\pi,M}_{s_{jp}}\right)_{1\le k,l\le M} &= \mathbb{E}\left[\psi^M_k(s_{jp}, X(s_{jp})) \psi^M_l(s_{jp}, X(s_{jp}))\right] \mathbb{E}\left[\Delta^2 W_{jp}\right], \qquad (5.4) \\ \left(\tilde{A}^{\pi,M}_{s_{jp}}\right)_{1\le k,l\le M} &= \mathbb{E}\left[\tilde{\psi}^M_k(s_{jp}, X(s_{jp})) \tilde{\psi}^M_l(s_{jp}, X(s_{jp}))\right] \mathbb{E}\left[\Delta^2 \tilde{N}_{jp}\right]. \qquad (5.5) \end{aligned}$$

Here, we define the process $Y^{\pi,M}(T)$ by setting $Y^{\pi,M}(T) = w(X(T))$, and then recursively

$$\begin{aligned} Y^{\pi,M}(s_{jp}) &= \alpha^{\pi,M}_{s_{jp}} m^M(s_{jp}, X(s_{jp})) \\ &\quad + g\left(s_{jp}, \gamma^{\pi,M}_{s_{jp}} \psi^M(s_{jp}, X(s_{jp})), \tilde{\gamma}^{\pi,M}_{s_{jp}} \tilde{\psi}^M(s_{jp}, X(s_{jp}))\right)\Delta_{jp}. \qquad (5.6) \end{aligned}$$

To compute the expectations in (5.2)-(5.5) numerically, we simulate N_0 independent paths $(X^n(s_{jp}))_{s_{jp}}$, starting with $X(T)$ for $s_{jp} = T$. Then, for $n = 1, \dots, N_0$, we define $y^{\pi,M,N_0}(T, x) := w(x)$, and

$$\begin{aligned} y^{\pi,M,N_0}(s_{jp}, x) :=& \alpha^{\pi,M,N_0}_{s_{jp}} m^M(s_{jp}, x) \\ &+ g\left(s_{jp}, \gamma^{\pi,M,N_0}_{s_{jp}} \psi^M(s_{jp}, x), \tilde{\gamma}^{\pi,M,N_0}_{s_{jp}} \tilde{\psi}^M(s_{jp}, x)\right)\Delta_{jp}, \qquad (5.7) \end{aligned}$$

where

$$\alpha_{s_{jp}}^{\pi,M,N_0} = \left(A_{s_{jp}}^{\pi,M,N_0}\right)^{-1} \frac{1}{N_0} \sum_{n=1}^{N_0} Y^{\pi,M,N_0}\left(s_{j(p+1)}\right) m^M(s_{jp}, X^n(s_{jp}))$$

$$\gamma_{s_{jp}}^{\pi,M,N_0} = \left(\bar{A}_{s_{jp}}^{\pi,M,N_0}\right)^{-1} \frac{1}{N_0} \sum_{n=1}^{N_0} Y^{\pi,M,N_0}\left(s_{j(p+1)}\right) \psi^M(s_{jp}, X^n())\Delta W_{jp}^n$$

$$\tilde{\gamma}_{s_{jp}}^{\pi,M,N_0} = \left(\tilde{A}_{s_{jp}}^{\pi,M,N_0}\right)^{-1} \frac{1}{N_0} \sum_{n=1}^{N_0} Y^{\pi,M,N_0}\left(s_{j(p+1)}\right) \tilde{\psi}^M(s_{jp}, X^n(s_{jp}))\Delta \tilde{N}_{jp}^n, \tag{5.8}$$

with coefficients given by

$$\left(A_{s_{jp}}^{\pi,M,N_0}\right)_{1\leq k,l\leq M} = \frac{1}{N_0} \sum_{n=1}^{N_0} m_k^M(s_{jp}, X^n(s_{jp}))m_l^M(s_{jp}, X^n(s_{jp}))$$

$$\left(\bar{A}_{s_{jp}}^{\pi,M,N_0}\right)_{1\leq k,l\leq M} = \frac{1}{N_0} \sum_{n=1}^{N_0} \psi_k^M(s_{jp}, X^n(s_{jp}))\psi_l^M(s_{jp}, X^n(s_{jp}))\Delta_{jp}$$

$$\left(\tilde{A}_{s_{jp}}^{\pi,M,N_0}\right)_{1\leq k,l\leq M} = \frac{1}{N_0} \sum_{n=1}^{N_0} \tilde{\psi}_k^M(s_{jp}, X^n(s_{jp}))\tilde{\psi}_l^M(s_{jp}, X^n(s_{jp}))\lambda^{\mathbb{P}}\Delta_{jp}. \tag{5.9}$$

We stop if $s_{jp} = 0$.

After that, we finally define $u^{h,\pi,M,N_0}(s_{jp}, x) := y^{h,\pi,M,N_0}(s_{jp}, x)$, $z^{h,\pi,M,N_0}(s_{jp}, x) := \gamma_{s_{jp}}^{h,\pi,M,N_0}\psi^M(s_{jp}, x)$ and $\tilde{z}^{h,\pi,M,N_0}(s_{jp}, x) := \tilde{\gamma}_{s_{jp}}^{h,\pi,M,N_0}\tilde{\psi}^M(s_{jp}, x)$.

Step (1.b.): Construct an Approximation to V^*

To do a backward recursion over t_j, we start by initializing $t_j = T$ and then set $V^{*,\pi}(T) = v^{*,\pi}(T, X(T)) := \Pi(T, X(T))$, and by assuming that we are given an approximation $V^{*,\pi,M,N_1}(t_{j+1}) = v^{*,\pi,M,N_1}(t_{j+1}, X(t_{j+1}))$, we carry out the following loop: For $p = P$, we initialize $U^\pi(s_{jP}) := U(t_{j+1}, V^{*,\pi,M,N_1}(t_{j+1})) = V^{*,\pi}(t_{j+1})$. Now, given $U^\pi\left(s_{j(p+1)}\right)$, $U^\pi(s_{jp})$ we know from (5.10) that

$$\begin{aligned} U^\pi(s_{jp}) \quad &\approx \quad U^\pi\left(s_{j(p+1)}\right) + g(s_{jp}, Z(s_{jp}), \tilde{Z}(s_{jp}))\left(s_{j(p+1)} - s_{jp}\right) \\ &\quad -Z(s_{jp})\Delta W_{j(p+1)} - \tilde{Z}(t_j)\Delta\tilde{N}_{j(p+1)}. \end{aligned}$$

Therefore, using N_1 simulations we can construct the vectors $(u^{\pi,M,N_1}(s_{jp}))_p$, $(\alpha_{s_{jp}}^{\pi,M,N_1})_p$, $(\gamma_{s_{jp}}^{\pi,M,N_1})_p$, and $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_1})_p$ (with $T = t_{j+1}$, $t_0 = t_j$ as before, and $w(\cdot) = v^{*,\pi}(t_{j+1}, \cdot)$ as terminal condition). This yields functions $u^{\pi,M,N_1}, z^{\pi,M,N_1}$ and $\tilde{z}^{\pi,M,N_1}$. Finally, when we have arrived at $p = 0$, we set $j = j - 1$ and by (5.2) we define

$$v^{*,\pi,M,N_1}(t_j, x) := \max(\Pi(t_j, x) + u^{h,\pi,M,N_1}(t_j, x), u^{\pi,M,N_1}(s_{j0}, x)).$$

We stop if $j = 0$.

Step (1.c.): Construct an Approximation to U^c

First note that the continuation value is requested on the coarse grid only. Let us consider the two points t_j, t_{j+1} on the coarse grid. Given the approximation of V^* from step (1.b.), basis functions $\phi^{\pi,M}(t_j, X(t_j)) = (\phi_1^{\pi}(t_j, X(t_j)), ..., \phi_M^{\pi}(t_j, X(t_j)))^T$ and coefficients $\beta_{t_j}^{\pi,M}$ we solve the regression

$$\beta_{t_j}^{\pi,M} = \arg\min_{\beta \in \mathbb{R}^M} \mathbb{E}\left[\left(U^{\pi}(t_j) - \beta^{\pi,M}\phi^{\pi,M}(t_j, X(t_j))\right)^2\right].$$

The approximated continuation value then reads

$$U^c(t_j, X(t_j)) = \beta_{t_j}^{\pi,M}\phi^{\pi,M}(t_j, X(t_j)).$$

Step (1.d.): Construct an Approximation to M^{*g}

We then obtain a martingale $M^{g,\pi,M,N_1}(s_{ip})$ by defining

$$\begin{aligned} M^{g,\pi,M,N_1}(s_{ip}) := &- \sum_{j=0}^{i}\sum_{l=0}^{p-1}\int_{s_{jl}}^{s_{j(l+1)}} g\left(u, z^{\pi,M,N_1}(s_{jl}, X(s_{jl})), \tilde{z}^{\pi,M,N_1}(s_{jl}, X(s_{jl}))\right) du \\ &+ \sum_{j=0}^{i}\sum_{l=0}^{p-1} z^{\pi,M,N_1}(s_{lp}, X(s_{jl}))\Delta W_{jl} + \sum_{j=0}^{i}\sum_{l=0}^{p-1} \tilde{z}^{\pi,M,N_1}(s_{lp}, X(s_{jl}))\Delta \tilde{N}_{jl}, \end{aligned} \tag{5.10}$$

see (5.11). Given i.i.d. simulations X^n we can then simulate i.i.d. copies of $M^{g,\pi}(s_{ip})$ through

$$\begin{aligned} M^{g,\pi,M,N_1,n}(s_{ip}) : = &- \sum_{j=0}^{i}\sum_{l=0}^{p-1}\int_{s_{jl}}^{s_{j(l+1)}} g\left(u, z^{\pi,M,N_1}(s_{jl}, X^n(s_{jl})), \tilde{z}^{\pi,M,N_1}(s_{jl}, X^n(s_{jl}))\right) du \\ &+ \sum_{j=0}^{i}\sum_{l=0}^{p-1} z^{\pi,M,N_1}(s_{lp}, X^n(s_{jl}))\Delta W_{jl}^n \\ &+ \sum_{j=0}^{i}\sum_{l=0}^{p-1} \tilde{z}^{\pi,M,N_1}(s_{lp}, X^n(s_{jl}))\Delta \tilde{N}_{jl}^n. \end{aligned} \tag{5.11}$$

Note that $(M^{g,\pi,M,N_1}(t_j))_{j\in\{0,1,2,...,L\}}$, defined by (5.10) is a true discrete-time U-martingale that (5.11) gives rise to an exact simulation scheme of it. The simulations $(M^{g,\pi,M,N_1,n})(t_j)$ will be employed to establish a dual upper bound to the Snell envelope and the simulations $(M^{g,\pi,M,N_1,n})(s_{jp})$ (living on the finer grid π) will be needed for the numerical approximation.

Step (2.): Construct an Approximated Upper Bound to V^*

Eventually (in Step (3.) below) we will find a genuine (biased high) upper bound for $V^*(0)$ according to Proposition 57. To this end, we are faced with the computation of

$$\begin{aligned} V^*(0) &= \inf_{M\in\mathcal{M}_0^U} U\left(0, \max_{t_j\in\{0,t_1,\ldots,T\}}\left(\Pi(t_j, X(t_j)) + U^h(t_j) + M(T) - M(t_j)\right)\right) \\ &= U\left(0, \max_{t_j\in\{0,t_1,\ldots,T\}}\left(\Pi(t_j, X(t_j)) + U^h(t_j) + M^{*g}(T) - M^{*g}(t_j)\right)\right). \end{aligned} \tag{5.12}$$

We set

$$F := \max_{t_j}\left(\Pi(t_j, X(t_j)) + U^h(t_j) + M^{*g}(T) - M^{*g}(t_j)\right).$$

Since we can only compute an approximation to M^{*g} we cannot attain the infimum in (5.12). However, $M^{g,\pi,M,N_1}(T)$ obtained in the previous Step (1.c.) is a true U-martingale, which can be used to obtain an approximation to an upper bound. Let us first define, with $N_0 = N_1$,

$$F^{\pi,M,N_1} := \max_{t_j\in\{0,t_1,\ldots,T\}}\left(\Pi(t_j, X(t_j)) + U^{h,\pi,M,N_1}(t_j) + M^{g,\pi,M,N_1}(T) - M^{g,\pi,M,N_1}(t_j)\right).$$

Next, define the $n+2$-dimensional Markov process

$$\begin{aligned} \mathcal{X}^{\pi,M,N_1}(s) &:= \Big(X(s_{jp}), M^{g,\pi,M,N_1}(s_{jp}), \\ &\qquad \max_{t_l\in\{0,t_1,\ldots,t_j\}}\left(\Pi(t_l, X(t_l)) + U^{h,\pi,M,N_1}(t_l) - M^{g,\pi,M,N_1}(t_l)\right)\Big) \end{aligned}$$

for $s_{jp} \le s < s_{j(p+1)}$. Let us compute $U\left(0, F^{\pi,M,N_1}\right)$ numerically. Recall that for a payoff H, by Theorem 59(a),

$$U(t,H) = \inf_{\mathbb{Q}\sim\mathbb{P}}\left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[H + \int_t^T r(s, q(s), \lambda(s) - \lambda^{\mathbb{P}})ds)\right]\right\} \tag{5.13}$$

$$= H + \int_t^T g\left(s, Z(s), \tilde{Z}(s)\right)ds - \int_t^T Z(s)\,dW(s) - \int_t^T \tilde{Z}(s)\,d\tilde{N}(s). \tag{5.14}$$

Hence, we can apply the approximation scheme (5.7)-(5.9) (with $X = \mathcal{X}$ and terminal condition $\max_{t_l\in\{0,t_1,\ldots,t_j\}}\left(\Pi(t_l, X(t_l)) + U^{h,\pi,M,N_1}(t_l) - M^{g,\pi,M,N_1}(t_l)\right)$). Simulate $n = 1,\ldots,N_2$ paths

$$\begin{aligned} \left(\mathcal{X}^{\pi,M,N_1,n}(s_{jp})\right)_j &= \Big(X^{\pi,n}(s_{jp}), M^{g,\pi,M,N_1,n}(s_{jp}), \\ &\qquad \max_{t_l\in\{0,t_1,\ldots t_j\}}\left(\Pi(t_l, X^n(t_l)) + U^{h,\pi,M,N_1}(t_l) - M^{g,\pi,M,N_1,n}(t_l)\right)\Big). \end{aligned}$$

Let M be the number of basis functions in the least squares Monte Carlo regression. We then obtain coefficients, let us say α_j^{π,M,N_2}, $\gamma_j^{\pi,M,N_2}, \tilde{\gamma}_j^{\pi,M,N_2}$, and processes $(V^{\pi,M,N_2}(t), Z^{\pi,M,N_2}(t), \tilde{Z}^{\pi,M,N_2}(t))_{0\leq t\leq T}$. Then, by applying Theorem 68 in the Section 5.5 twice, we may conclude that

$$\lim_{\pi\to 0}\lim_{M\to\infty}\lim_{N_2,N_1\to\infty}\left(V^{\pi,M,N_2}, Z^{\pi,M,N_2}, \tilde{Z}^{\pi,M,N_2}\right) = \left(V^*, Z^*, \tilde{Z}^*\right); \tag{5.15}$$

see the technical details provided in the Section 5.5. In particular, $V^{\pi,M,N_2}(0) \to V(0)$ as the mesh ratio of the grid, π, tends to zero, and the number of Monte Carlo simulations and basis functions tend to infinity. Thus, our algorithm will converge to the *true* value of the (U-)Snell envelope V^*.

However, at the pre-limiting level, our estimates from Step (2.) for the upper bound to V^* are not biased high (above the Snell envelope), meaning that in the average the upper bound may not provide enough protection. For this reason we will subsequently proceed to construct a genuine (biased high) upper bound.

Step (3.a.): Construct a Genuine Upper Bound to U^h

By Theorem 59(a), for $i = 0, \ldots, L-1$,

$$U^h(t_i) = \inf_{\mathbb{Q}\sim\mathbb{P}}\left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_{t_i}}\left[\sum_{j=i}^{L} h(t_j, X(t_j)) + \int_{t_i}^{T} r\left(s, q(s), \lambda(s) - \lambda^{\mathbb{P}}\right) ds\right]\right\}, \tag{5.16}$$

$$\begin{aligned} &= U^h(t_{i+1}) + \int_{t_i}^{t_{i+1}} g\left(s, Z^h(s), \tilde{Z}^h(s)\right) ds \\ &- \int_{t_i}^{t_{i+1}} Z^h(s)\, dW(s) - \int_{t_i}^{t_{i+1}} \tilde{Z}^h(s)\, d\tilde{N}(s) + h(t_i, X(t_i)). \end{aligned} \tag{5.17}$$

Denote the $\mathbb{Q}$ that attains the infimum in (5.16) by $\mathbb{Q}^h$.

The following proposition provides a way to practically obtain the extremal $\mathbb{Q}^h$ (leading in the end to an upper bound) by computing $(Z^h, \tilde{Z}^h)$ in (5.17).

Proposition 65 *For $H \in L^2(\mathbb{P})$, the infimum in (5.16) is attained in*

$$\frac{d\mathbb{Q}^h}{d\mathbb{P}} = \exp\left\{\int_0^t q^*(s)\, dW_s + \int_0^t \log\left(\frac{\lambda^*(s)}{\lambda^{\mathbb{P}}}\right) d\tilde{N}(s) - \int_0^T \left(\frac{\|q^*(s)\|^2}{2} + \lambda^{\mathbb{P}} - \lambda^*(s)\right) ds\right\},$$

for every $(q^*(s), \lambda^*(s) - \lambda^{\mathbb{P}}) \in \partial g(s, Z^h(s), \tilde{Z}^h(s))$, where $\partial g(s, \cdot, \cdot)$ stands for the subdifferentials of the convex function $g(s, \cdot, \cdot)$.[2]

[2] Formally, $\partial f(x)$ of a convex function is given by the set of all slopes of all tangents at $f(x)$. Of course, in the one-dimensional case, $\partial f(x) = [f_-(x), f'_+(x)]$. Furthermore, $\partial f(x) = \{f'(x)\}$ if f is differentiable.

We then compute a genuine upper bound to $(U^h(t_j))_{t_j\in\{0,\ldots,t_{L-1}\}}$ by:

(i) Computing approximations to $(Z,\tilde{Z})$ by solving (5.17). In view of Proposition 65, $(Z,\tilde{Z})$ induce an approximation to $\mathbb{Q}^h$, say $\mathbb{Q}^{h,\text{approx}}$.

(ii) Evaluating $\mathbb{E}^{X(t_i)}_{\mathbb{Q}^{h,\text{approx}}}\big[\sum_{j=i}^L h(t_j,X(t_j))+\int_{t_i}^T r(s,q(s),\lambda(s)-\lambda^{\mathbb{P}})ds\big]$ and making use of (5.16), which will deliver the desired genuine (biased high) upper bound to $(U^h(t_j))_{t_j\in\{0,\ldots,t_{L-1}\}}$.

So let us carry out our program to compute approximations of the form $U^{h,n}(t_j)=u^{h,n}(t_j,X^n(t_j))$, for $n=1,\ldots,N_3$: Simulate N_3 copies of $(X^n(s_{jp}))$ ("outer simulation"). For $X^n(t_j)=x$, let $N_4\in\mathbb{N}$ and simulate additional paths $(X^{t_j,x,n}(s_{jp}))$ for $n=1,\ldots,N_4$ and j,p ("inner simulation"). For simplicity, assume that $g(s,\cdot,\cdot)$ is continuously differentiable. If this is not the case, then our arguments may still be seen to hold by taking elements in the subgradient. Define, with $N_0=N_1$, $z^{h,\pi,M,N_1}(s_{jp},\bar{x}):=\gamma^{h,\pi,M,N_1}_{s_{jp}}\psi^M(s_{jp},\bar{x})$, $\tilde{z}^{h,\pi,M,N_1}(s_{jp},\bar{x}):=\tilde{\gamma}^{\pi,M,N_1}_{s_{jp}}\tilde{\psi}^M(s_{jp},\bar{x})$ and

$$q^{h,\pi,t_j,x,n}(s_{jp}):=g_z\left(s_{jp},z^{h,\pi,M,N_1}\left(s_{jp},X^{t_j,x,n}(s_{jp})\right),\tilde{z}^{h,\pi,M,N_1}\left(s_{jp},X^{t_j,x,n}(s_{jp})\right)\right)$$
$$\lambda^{h,\pi,t_j,x,n}(s_{jp})-\lambda^{\mathbb{P}}:=g_{\tilde{z}}\left(s_{jp},z^{h,\pi,M,N_1}\left(s_{jp},X^{t_j,x,n}(s_{jp})\right),\tilde{z}^{h,\pi,M,N_1}\left(s_{jp},X^{t_j,x,n}(s_{jp})\right)\right).$$

Next, for each $X^n(t_j)=x$, define i.i.d. simulations of the measure $\frac{d\mathbb{Q}^{g,\pi,M,N_4,t_j,x}}{d\mathbb{P}}$ via the Radon-Nikodym derivative

$$\begin{aligned}D^{\pi,n}(t_i,x):=\exp\Bigg(&\sum_{t_i\le s_{jp}}q^{\pi,t_j,x,n}(s_{jp})\,\Delta W^n_{jp}+\sum_{t_i\le s_{jp}}\log\left(\frac{\lambda^{\pi,t_j,x,n}(s_{jp})}{\lambda^{\mathbb{P}}}\right)\Delta\tilde{N}^n_{jp}\\&-\sum_{t_i\le s_{jp}}\left(\frac{1}{2}\left\|q^{\pi,t_j,x,n}(s_{jp})\right\|^2+\lambda^{\mathbb{P}}-\lambda^{\pi,t_j,x,n}(s_{jp})\right)\Delta_{jp}\Bigg);\end{aligned}$$

for $i=1,\ldots,L$, see also (5.5). We then set

$$\begin{aligned}\widetilde{u}^{upp,h,N_4}(t_j,x):=&\frac{1}{N_4}\sum_{n=1}^{N_4}D^{\pi,n}(t_j,x)\left[\sum_{l=j}^{L}h(t_l,X^{t_j,x,n}(t_l))\right.\\&\left.+\sum_{l=1}^{L}\sum_{p=1}^{P}\int_{s_{lp}}^{s_{l(p+1)}}r(s,q^{h,\pi,t_l,x,n}(s_{lp}),\lambda^{h,\pi,t_l,x,n}(s_{lp})-\lambda^{\mathbb{P}})ds\right].\end{aligned}\tag{5.18}$$

Now $(D^{\pi,n}(t_j,X^n(t_j)))$, $(q^{h,\pi,M,N_4,n}(s_{jp}))_{j,p}$ and $(\lambda^{h,\pi,M,N_4,n}(s_{jp}))_{j,p}$ are true i.i.d. simulations of $\frac{d\mathbb{Q}^{h,\pi,M,N_4}}{d\mathbb{P}}$, the piecewise constant $(q(t))$ and $(\lambda(t))$, conditioned on $X(t_j)=x$. Therefore, by (5.14), $\widetilde{u}^{upp,h,N_4}(t,X^n(t))$ can be taken as approximative simulations of $U^h(t)$, yielding a genuine (biased high) upper bound to $U^h(t)=u^h(t,X(t))$. Summarizing this step, we obtain the following proposition.

Proposition 66 *We have* $\mathbb{E}\left[\widetilde{u}^{upp,h,\pi}(t,x)\right] \geq u^h(t,x)$*, for any* x*.*

Step (3.b.): Construct a Genuine Upper Bound to $V^*(0)$

In this final step, we proceed as in Step (3.a.) above, but this time we only need to compute an upper bound at time $t = 0$: Denote the $\mathbb{Q}$ that attains the infimum in (5.13) by $\mathbb{Q}^g$, with corresponding $(q^*(s), \lambda^*(s) - \lambda^{\mathbb{P}})$. As in Proposition 65 one may see that $(q^*(s), \lambda^*(s) - \lambda^{\mathbb{P}}) \in \partial g(s, Z(s), \tilde{Z}(s))$ with $(Z, \tilde{Z})$ from (5.14). We shall exploit this to practically compute our approximation. Let $N_3 \in \mathbb{N}$ and simulate paths $(W^n(s_{jp}))$ and $(X^n(s_{jp}))$ for $n = 1, \ldots, N_3$ and j, p. Define

$$U^{upper,h,\pi,n}(t_j) := \tilde{u}^{upp,h,\pi}(t_j, X^n(t_j)),$$

$$q^{\pi,M,N_2,n}(s_{jp}) := g_z\left(s_{jp}, z^{\pi,M,N_2,n}(s_{jp}, X^n), \tilde{z}^{\pi,M,N_2,n}(s_{jp}, X^n)\right)$$

$$\lambda^{\pi,M,N_2,n}(s_{jp}) - \lambda^{\mathbb{P}} := g_{\tilde{z}}\left(s_{jp}, z^{\pi,M,N_2,n}(s_{jp}, X^n), \tilde{z}^{\pi,M,N_2,n}(s_{jp}, X^n)\right).$$

Next, define i.i.d. simulations $\frac{d\mathbb{Q}^{g,\pi,M,N_3,n}}{d\mathbb{P}}$ via

$$\frac{d\mathbb{Q}^{g,\pi,M,N_3,n}}{d\mathbb{P}} := \exp\left(\sum_{j,p} q^{\pi,M,N_2,n}(s_{jp})\Delta W^n_{jp} + \sum_{j,p}\log\left(1 + \frac{\lambda^{\pi,M,N_2,n}(s_{jp})}{\lambda^{\mathbb{P}}}\right)\Delta\tilde{N}^n_{jp}\right.$$
$$\left. - \sum_{j,p}\left(\frac{1}{2}\left\|q^{\pi,M,N_2,n}(s_{jp})\right\|^2 + \lambda^{\mathbb{P}} - \lambda^{\pi,M,N_2,n}(s_{jp})\right)\Delta_{jp}\right),$$

Finally, we set

$$\widetilde{V}^{upp,N_3}(0) := \frac{1}{N_3}\sum_{n=1}^{N_3}\frac{d\mathbb{Q}^{g,\pi,M,N_3,n}}{d\mathbb{P}}$$
$$\left[\max_{t_j\in\{0,\ldots,T\}}\left(\Pi(t_j, X^n(t_j)) + U^{upper,h,\pi,n}(t_j) + M^{g,\pi,M,N_1,n}(T) - M^{g,\pi,M,N_1,n}(t_j)\right)\right.$$
$$\left. + \sum_{j,p}\int_{s_{jp}}^{s_{j(p+1)}} r(s, q^{\pi,M,N_2,n}(s_{jp}), \lambda^{\pi,M,N_2,n}(s_{jp}) - \lambda^{\mathbb{P}})ds\right], \tag{5.19}$$

where $M^{g,\pi,M,N_1,n}(t_j)$ should be simulated using α^{π,M,N_1}, γ^{π,M,N_1} and $\tilde{\gamma}^{\pi,M,N_1}$ estimated previously (under Step (1.)).

Step (4.): Construct a lower bound to $V^*(0)$

We follow the usual approach by constructing a stopping time τ depending on the cash flow and the continuation value. Clearly the problem is to obtain a good approximation for the continuation value which is given by

$$C(t_j) = U(t_j, V^*(t_{j+1}))$$

Our approximation $c(t_j)$ will be obtained via regression. Therefore consider step (1.b.) where BSDE $(U(t), Z(t))$ is solved. First note that the continuation value is requested on the coarse grid only. Let us consider the two points $t_j,\ t_{j+1}$ on the coarse grid. Given terminal condition $U(t_{j+1}, V^*(t_{j+1})) = V^*(t_{j+1})$ the BSDE $(U(t), Z(t))$ is solved up to time $s_{j_0} = t_j$. Given a partition π, basis functions $\phi^{\pi,M}(t_j, X(t_j)) = (\phi_1^{\pi}(t_j, X(t_j)), ..., \phi_M^{\pi}(t_j, X(t_j)))^T$ and coefficients $\beta_{t_j}^{\pi,M}$ we solve the regression

$$\beta_{t_j}^{\pi,M} = \arg\min_{\beta\in\mathbb{R}^M} \mathbb{E}\left[\left(U(t_j) - \beta^{\pi,M}\phi^{\pi,M}(t_j, X(t_j))\right)^2\right].$$

The approximated continuation value then reads

$$u^{c,\pi}(t_j)(X(t_j)) = \beta_{t_j}^{\pi,M}\phi^{\pi,M}(t_j, X(t_j)).$$

To compute the lower bound we define a stopping rule by

$$\tau(t_j) := \inf_{t\in\{t_j,...,T\}}\left\{\Pi(t, X(t)) + U^h(t) \geq U^c(t)\right\}, \quad t_j \in \{0, ..., T\}.$$

Given $N_5 \in \mathbb{N}$ independently simulated trajectories we define

$$\tau^n(t_j) := \inf_{t\in\{t_j,...,T\}}\left\{\Pi(t, X^n(t)) + U^{h,n}(t) \geq \beta_t^{\pi,M}\phi^{\pi,M}(t, X^n(t))\right\}, \quad t_j \in \{0, ..., T\}.$$

We then face the problem of computing a solution to a BSDE with random terminal time. This problem was, among others, studied by Briand and Confortola [2008], Briand and Hu [2008] and Darling and Pardoux [2008]. Such a BSDE can be written in the following form

$$Y(t\wedge\tau) = \xi + \int_{t\wedge\tau}^{T\wedge\tau} g(s, Z(s))\,ds - \int_{t\wedge\tau}^{T\wedge\tau} Z(s)\,dW(s), \quad \forall 0 \leq t \leq T,$$

where ξ is the terminal condition at the random time τ. For the set $\{t > \tau\}$ we have $Y(t) = \xi$ and $Z_t = 0$. From the implementation point of view we solve the BSDE as done previously but regress only over trajectories that were not already stopped. As there are no references for solving BSDEs with random terminal time by regression I am not quite sure whether this is correct . To clarify, trajectories already stopped may have an impact to the regression procedure but this impact does not depend on the current state of the underlying process on such a trajectory. We should discuss this point. For trajectory n at the pre-computed stopping time $\tau^{(n)}$ we initialize the BSDE with the current payoff given by $\Pi(\tau, X(\tau))$ and then solve in the standard way.

Summary and Main Result

Let us summarize our algorithm more succinctly. Given a fixed time grid π and M basis functions:

(1.) Run N_0 Monte Carlo simulations to compute U^{h,π,M,N_0} and U^{c,π,M,N_0}. Run Monte Carlo N_1 simulations to compute M^{g,π,M,N_1}. To fully describe the evolution of these processes, it is sufficient to store the corresponding $(\alpha_{s_{jp}}^{h,\pi,M,N_0})$, $(\gamma_{s_{jp}}^{h,\pi,M,N_0})$, $(\tilde{\gamma}_{s_{jp}}^{h,\pi,M,N_0})$, $(\beta_{t_j}^{\pi,M,N_0})$; and $(\alpha_{s_{jp}}^{\pi,M,N_1})$, $(\gamma_{s_{jp}}^{\pi,M,N_1})$, $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_1})$ and , $(\phi_{t_j}^{\pi,M,N_0})$.

(2.) With $N_0 = N_1$, $(\alpha_{s_{jp}}^{h,\pi,M,N_1})$, $(\gamma_{s_{jp}}^{h,\pi,M,N_1})$, $(\tilde{\gamma}_{s_{jp}}^{h,\pi,M,N_1})$ as well as $(\alpha_{s_{jp}}^{\pi,M,N_1})$, $(\gamma_{s_{jp}}^{\pi,M,N_1})$, $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_1})$ give rise to a terminal condition F^{π,M,N_1} and a Markov process $\mathcal{X}^{\pi,M,N_1}$ defined under Step (2.). Run N_2 Monte Carlo simulations to calculate $(V^{\pi,M,N_2}, Z^{\pi,M,N_2}, \tilde{Z}^{\pi,M,N_2})$ as the solution to corresponding BSΔEs with the Markov process $\mathcal{X}^{\pi,M,N_1}$ and terminal condition F^{π,M,N_1}. Store the corresponding $(\gamma_{s_{jp}}^{\pi,M,N_2})$ and $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_2})$.

(3.a.) Simulate N_3 (outer simulation) copies of $(X^n(s_{jp}))$. Simulate, for every n, j, p, N_4 additional (inner simulation) copies of $(X^n(s_{jp}))$, to eventually compute, with $(\gamma_{s_{jp}}^{h,\pi,M,N_1})$ and $(\tilde{\gamma}_{s_{jp}}^{h,\pi,M,N_1})$ at hand from the previous Step (1.), N_3 copies of $U^{upper,h,n}$.

(3.b.) With $(\gamma_{s_{jp}}^{\pi,M,N_2})$ and $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_2})$ at hand from the previous Step (2.), simulate N_3 copies of $\frac{d\mathbb{Q}^{g,\pi}}{d\mathbb{P}}$. Furthermore, with $(\alpha_{s_{jp}}^{\pi,M,N_1})$, $(\gamma_{s_{jp}}^{\pi,M,N_1})$ and $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_1})$ at hand from the previous Step (1.), simulate N_3 copies of F^{π,M,N_1}. Using (5.19), a genuine (biased high) estimate for V^* can then be obtained.

(4.) Simulate N_5 (outer simulation) copies of $(X^n(s_{jp}))$. Compute, with $(\beta_{t_j}^{\pi,M,N_5})$ at hand from Step (1.), N_5 copies of $U^{c,n}$ and extract stopping times $\tau^n(t_j) = \inf_{t\in\{t_j,\dots,T\}} \left\{\Pi(t, X^n(t)) + U^{h,n}(t) \geq \beta_t^{\pi,M} \phi^{\pi,M}(t, X^n(t))\right\}$. Use these stopping times to calculate the corresponding BSDE with random terminal condition $\Pi(\tau^n, X(\tau^n)) + U^h(\tau^n)$.

Our main result, then, reads as follows:

Theorem 67 *Under (G1)-(G3) the primal estimator $V^{\pi,M,N_1}(0)$ and both the dual estimators $V^{\pi,M,N_2}(0)$ and $\widetilde{V}^{upp,N_3}(0)$ converge to the upper Snell envelope, i.e.,*

$$\begin{aligned}
\lim_{\pi\to 0}\lim_{M\to\infty}\lim_{N_1\to\infty} V^{\pi,M,N_1}(0) &= \lim_{\pi\to 0}\lim_{M\to\infty}\lim_{N_2,N_1\to\infty} V^{\pi,M,N_2}(0) \\
&= \lim_{\pi\to 0}\lim_{M\to\infty}\lim_{N_3,N_2,N_1\to\infty} \widetilde{V}^{upp,N_3}(0) = V^*(0).
\end{aligned}$$

Furthermore, with $(\alpha_{s_{jp}}^{\pi,M,N_1})$, $(\gamma_{s_{jp}}^{\pi,M,N_1})$, $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_1})$, $(\gamma_{s_{jp}}^{\pi,M,N_2})$ and $(\tilde{\gamma}_{s_{jp}}^{\pi,M,N_2})$ fixed from the preceding Steps (1.) and (2.), our estimator in Step (3.) gives rise to a genuine (biased high) upper bound, i.e., $\mathbb{E}\left[\widetilde{V}^{upp,N_3}(0)\right] \geq V^*(0)$.

5.4. Numerical experiments

We present here our numerical results applying our algorithm to different examples of ambiguity. Consider assets X_i driven by

$$\frac{dX_i(t)}{X_i(t)} = \mu dt + \sigma dW_i(t) + Jd\tilde{N}_i(t), \quad X_i(0) = x_0, \quad i = 1, 2,$$

where $W_i(t)$ are independent one-dimensional Brownian motions and σ denotes the volatility. $\tilde{N}_i(t)$ are one-dimensional compensated Poisson processes with intensity $\lambda^{\mathbb{P}}$ and J denoting the jump size. $W_i(t)$ and $\tilde{N}_i(t)$ are assumed to be independent, too and in the case of multiple assets these are assumed to be uncorrelated. The drift μ is given by $r - \delta$, where r is the risk-free rate and δ denotes dividend payments. We deal with Bermudan max-call, call and put options with strike K. The discounted payoffs when exercising at time t are given by

$$\begin{aligned} e^{-rt}\left(\max\{X_1(t), X_2(t)\} - K\right)^+ &\quad - \quad \text{max-call}, \\ e^{-rt}\left(X(t) - K\right)^+ &\quad - \quad \text{call}, \\ e^{-rt}\left(K - X(t)\right) &\quad - \quad \text{put}. \end{aligned}$$

We write $X(t) = (X_1(t), X_2(t))$ in the two-dimensional and $X(t) = X_1(t)$ in the one-dimensional case.

As stated in section (5.3) our regression approach is based on the minimization of the form

$$\begin{aligned} \left[\alpha^{\pi,M}_{s_{jp}}, \gamma^{\pi,M}_{s_{jp}}\right] &= \arg\min_{\alpha,\gamma\in\mathbb{R}^M} \mathbb{E}\left[\left(Y^{\pi,M}\left(s_{j(p+1)}\right) - \sum_{m=1}^{M} \alpha_{s_{jp}} m^M_{s_{jp}}\left(s_{jp}, X\left(s_{jp}\right)\right)\right.\right. \\ &\quad - \sum_{m=1}^{M} \gamma_{s_{jp}} \psi^M_{s_{jp}}\left(s_{jp}, X\left(s_{jp}\right)\right)\left(W\left(s_{j(p+1)}\right) - W\left(s_{jp}\right)\right) \\ &\quad \left.\left. - \sum_{m=1}^{M} \tilde{\gamma}_{s_{jp}} \tilde{\psi}^M_{s_{jp}}\left(s_{jp}, X\left(s_{jp}\right)\right)\left(\tilde{N}_{\left(s_{j(p+1)}\right)} - \tilde{N}\left(s_{jp}\right)\right)\right)^2\right]. \end{aligned}$$

The choice of basis functions is crucial to obtain tight upper bounds. We will state them in detail for the different example . We further consider lower bounds calculated with respect to stopping times given by

$$\tau(t_j) := \inf_{t\in\{t_j,\ldots,T\}} \left\{\Pi(t, X(t)) + U^h(t) \geq U^c(t)\right\}, \quad t_j \in \{0, \ldots, T\}.$$

The values are presented in parenthesis in the corresponding tables. The choice of basis functions for ϕ is discussed in detail for the different examples, too.

5.4.1. Geometric Brownian motion

Let us first consider the situation without jumps resulting in $J = \lambda^{\mathbb{P}} = 0$. We will give examples regarding the uni- and the bivariate case. Following Schoenmakers, Zhang and Huang [2013] we look at the following parameter setting with respect to the reference measure $\mathbb{P}$:

$$r = 0.05,\ \delta = 0.1,\ \sigma = 0.2,\ K = 100,\ T = 3 \text{ yrs.}$$

Furthermore we have exercising dates given by $t_j = \frac{jT}{9}$, $j = 0, ..., 9$ and a fine grid $\{s_{jp}\}$ with $\Delta = 1/150$. For the choice of basis functions we more or less follow Schoenmakers, Zhang and Huang [2013] including still-alive European options and corresponding option deltas.

Univariate case

Let for now $E_\Pi(t, X(t), T)$ denote the value of a call option at time t with maturity T and $\frac{\partial E_\Pi(t,X(t),T)}{\partial X(t)}$ its derivative with respect to the underlying. For our algorithm we choose for $t_j \leq t \leq t_{j+1}$ the set of

$$\{1,\ \mathrm{Pol}_2(X(t)),\ \mathrm{Pol}_3(E_\Pi(t, X(t), t_{j+1})),\ \mathrm{Pol}_3(E_\Pi(t, X(t), t_L))\}$$

for m_t^M. For the Brownian motion driven part of the BSDE we choose for ψ_t^M the set

$$\left\{1,\ X(t)\frac{\partial E_\Pi(t, X(t), t_{j+1})}{\partial X(t)},\ X(t)\frac{\partial E_\Pi(t, X(t), t_L)}{\partial X(t)}\right\}.$$

$\mathrm{Pol}_n(y)$ denotes the set of monomials up to degree n of a vector y. The results presented here are based on 1000 trajectories for step 1b, so the estimation of the regression coefficients, and 5000 trajectories for the actual calculation of the $U-$martingale increments in step 1c and the upper bound in step 2. For step 2 ψ_t^M is enlarged by the martingale and maximum process as given in the process $\mathcal{X}$. For ϕ_t^M we choose the same set as for m_t^M.

Kullback-Leibler divergence

In Table 5.1 we consider the case of Kullback-Leibler divergence for different choices of α where the last column corresponds to the case of standard expectation.

The last column has to be interpreted as $g = 0$, so it corresponds to the case of standard conditional expectation.

Worst case with mean partially known

We here consider worst case with mean partially known where we fix $\mu^- = \mu$ in Table 5.2 and $\mu^+ = \mu$ in Table 5.3. We choose B^+ and B^- such that the driver is practically independent of these parameters, e.g. $B^+ = 1000$ and $B^+ = -1000$.

Comparing to Table 5.1 we conclude that the case of $\mu = \mu^+ = \mu^-$ gives the case of standard expectation. Note further that the worst case with mean partially known

x_0	$\alpha = 5$	$\alpha = 10$	$\alpha = 100$	$\alpha = 10^4$	$\alpha = 10^6$	$\alpha = \infty$
90	1.7463	2.4454	4.0505	4.4052	4.4091	4.4092
	(1.5368)	(2.3333)	(3.9978)	(4.3584)	(4.3625)	(4.3625)
100	3.3282	4.6440	7.4309	8.0041	8.0105	8.0105
	(3.0368)	(4.5303)	(7.3486)	(7.9365)	(7.9433)	(7.9434)
110	10.0087	10.1196	12.4255	13.1913	13.1999	13.2000
	(10.0000)	(10.0000)	(12.3073)	(13.0881)	(13.0974)	(13.0975)

Table 5.1.: Kullback-Leibler divergence driver for different α - univariate case

x_0	$\mu^+ = 0.95$, $\mu^- = -0.05$	$\mu^+ = 0.05$, $\mu^- = -0.05$	$\mu^+ = 0$, $\mu^- = -0.05$	$\mu^+ = -0.05$, $\mu^- = -0.05$
90	4.4188	4.4100	4.4096	4.4092
	(4.3348)	(4.3583)	(4.3604)	(4.3625)
100	8.0196	8.0114	8.0110	8.0105
	(7.9037)	(7.9381)	(7.9406)	(7.9434)
110	13.2007	13.2000	13.2000	13.2000
	(13.0627)	(13.0927)	(13.0950)	(13.0975)

Table 5.2.: worst case with mean partially known driver for $\mu^- = -0.05$ and different choices of μ^+ - univariate case

x_0	$\mu^+ = -0.05$, $\mu^- = -1.05$	$\mu^+ = -0.05$, $\mu^- = -0.15$	$\mu^+ = -0.05$, $\mu^- = -0.1$	$\mu^+ = -0.05$, $\mu^- = -0.05$
90	0.8343	1.6365	2.6194	4.4092
	(0.0000)	(1.4605)	(2.5213)	(4.3625)
100	0.9668	4.1001	5.5695.5691	8.0105
	(0.0000)	(3.8513)	(5.4314)	(7.9434)
110	9.992	10.1693	10.8396	13.2000
	(10.0000)	(10.0000)	(10.3407)	(13.0975)

Table 5.3.: worst case with mean partially known driver for $\mu^+ = -0.05$ and different choices of μ^- - univariate case

example in the setting without jumps equals the worst case with ball scenarios example for $\delta = \left|\frac{\mu^- - \mu}{\sigma}\right| = \left|\frac{\mu^+ - \mu}{\sigma}\right|$ subject to $\frac{\mu^+ - \mu}{\sigma} \leq B^+$ and $\frac{\mu^- - \mu}{\sigma} \geq B^-$.

Bivariate case

For the bivariate case we consider a max-call option and again denote its value at time t with maturity T by $E_\Pi\left(t, X\left(t\right), T\right)$. It is given by the following formula (Johnson [1987])

$$
\begin{aligned}
E_\Pi\left(t, X\left(t\right), T\right) &= \sum_{l=1}^{2} X_l\left(t\right) \frac{e^{-\delta(T-t)}}{\sqrt{2\pi}} \int\limits_{\left(-\infty, d_+^l\right]} \exp\left[-\frac{1}{2}z^2\right] \cdot \\
&\quad \cdot \prod_{l'=1, l'\neq l}^{2} \mathcal{N}\left(\frac{\ln\left(\frac{X_l(t)}{X_{l'}(t)}\right)}{\sigma\sqrt{T-t}} - z + \sigma\sqrt{T-t}\right) dz \\
&\quad - Ke^{-r(T-t)} + Ke^{-r(T-t)} \prod_{l=1}^{2}\left(1 - \mathcal{N}\left(d_-^l\right)\right)
\end{aligned}
$$

with

$$
d_-^l := \frac{\ln\left(\frac{X_l(t)}{K}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)\left(T-t\right)}{\sigma\sqrt{T-t}}, \quad d_+^l := d_-^l + \sigma\sqrt{T-t}.
$$

Here $\mathcal{N}$ denotes the standard Gaussian cumulative distribution. For the option delta, denoted by $\frac{\partial E_\Pi(t, X(t), T)}{\partial X_d(t)}$, it follows that

$$
\begin{aligned}
\frac{\partial E_\Pi\left(t, X\left(t\right), T\right)}{\partial X_d\left(t\right)} &= \frac{e^{-\delta(T-t)}}{\sqrt{2\pi}} \\
&\quad \int\limits_{\left(-\infty, d_+^d\right]} \exp\left[-\frac{1}{2}z^2\right] \prod_{l=1, l\neq d}^{2} \mathcal{N}\left(\frac{\ln\left(\frac{X_d(t)}{X_l(t)}\right)}{\sigma\sqrt{T-t}} - z + \sigma\sqrt{T-t}\right) dz.
\end{aligned}
$$

For step 1b of our algorithm we choose the same set of basis functions for m_t^M and ϕ_t^M as for the univariate case. For the Brownian motion driven part we have to adapt to the two-dimensionality by considering for ψ_t^M the set

$$
\left\{1, \left(X_d\left(t\right) \frac{\partial E_\Pi\left(t, X\left(t\right), t_{j+1}\right)}{\partial X_d\left(t\right)}\right)_{1\leq d\leq 2}, \left(X_d\left(t\right) \frac{\partial E_\Pi\left(t, X\left(t\right), t_L\right)}{\partial X_d\left(t\right)}\right)_{1\leq d\leq 2}\right\}.
$$

Again step 1b is based on 1000 trajectories, while taking 5000 trajectories for step 1c and 2 and enlarging ψ_t^M in step 2 by the martingale and maximum process as given in the process $\mathcal{X}$. The specific parameters are chosen as in the univariate case.

Kullback-Leibler

We consider the Kullback-Leibler divergence driver for different α.

x_0	$\alpha = 5$	$\alpha = 10$	$\alpha = 100$	$\alpha = 10^4$	$\alpha = 10^6$	$\alpha = \infty$
90	3.0909	4.4892	7.4893	8.1219	8.1219	8.1220
	(2.9140)	(4.3825)	(7.3874)	(8.0284)	(8.0284)	(8.0285)
100	5.5555	8.1205	12.9791	13.9669	13.9778	13.9779
	(5.4828)	(8.0509)	(12.8265)	(13.8248)	(13.8372)	(13.8374)
110	10.6160	13.4051	20.1193	21.4305	21.4451	21.4453
	(10.0000)	(13.4173)	(19.9179)	(21.2436)	(21.2589)	(21.2591)

Table 5.4.: Kullback-Leibler divergence driver for different α - bivariate case

The last column is given by considering the standard conditional expectation case, so $g = 0$, where we have reference values given in Andersen and Broadie [2004] (A&B) and Belomestny, Bender and Schoenmakers [2009] (BBS) as stated in Table 5.5 and Table 5.6.

Worst case with mean partially known

Let us now consider the situation of worst case with mean partially known. For different choices of μ^+ and μ^- upper bounds for the option price are given in the following Tables.

x_0	$\mu^+ = 0.95$, $\mu^- = -0.05$	$\mu^+ = 0.05$, $\mu^- = -0.05$	$\mu^+ = -0.05$, $\mu^- = -0.05$	BBS	A&B
90	8.1116	8.1228	8.1220	8.0891	8.082
	(7.8752)	(8.0038)	(8.0285)		
100	13.9615	13.9828	13.9779	13.958	13.934
	(13.6239)	(13.7989)	(13.8374)		
110	21.4228	21.4519	21.4453	21.459	21.359
	(20.9942)	(21.217)	(21.2591)		

Table 5.5.: worst case with mean partially known driver for $\mu^- = -0.05$ and different choices of μ^+ - bivariate case

x_0	$\mu^+ = -0.05$, $\mu^- = -1.05$	$\mu^+ = -0.05$, $\mu^- = -0.15$	$\mu^+ = -0.05$, $\mu^- = -0.05$	BBS	A&B
90	0.6836	3.0190	8.1220	8.0891	8.082
	(0.0000)	(2.7231)	(8.0285)		
100	0.6194	7.0795	13.9779	13.958	13.934
	(0.0000)	(6.7247)	(13.8374)		
110	9.9823	13.5768	21.4453	21.459	21.359
	(10.0000)	(13.2628)	(21.2591)		

Table 5.6.: worst case with mean partially known driver for $\mu^+ = -0.05$ and different choices of μ^- - bivariate case

The results in Table 5.5 and Table 5.6 for $\mu^+ = \mu^- = -0.05$ can be seen as the standard conditional expectation case as we have $g = 0$ for this choice. The calculation of a particular result in one of the Tables took approximately 18 minutes as these calculations among other things involve multiple numerical integrations for the calculation of the basis functions.

5.4.2. Geometric Brownian motion with jumps

Let us now consider the situation of one asset driven by a geometric Brownian motion with jumps. For the jump size we choose $J = 0.06$, while considering different choices for the jump intensity $\lambda^{\mathbb{P}}$. We use the following parameter setting

$$r = 0.04,\ \delta = 0,\ \sigma = 0.2,\ K = 100,\ T = 1 \text{ yr},$$

where exercising dates are given by $t_j = \frac{jT}{10}$, $j = 0, ..., 10$. For the fine grid we work with $\Delta = 1/100$.

Let here $E_\Pi\left(t, X\left(t\right), T\right)$ denote the value of a put option at time t with maturity T and $\frac{\partial E_\Pi(t,X(t),T)}{\partial X(t)}$ its derivative with respect to the underlying. Following Cont and Tankov [2004] the expected payoff is given by

$$E_\Pi\left(t, X\left(t\right), T\right) = e^{-r(T-t)} \sum_{n \geq 0} \frac{e^{-\lambda^{\mathbb{P}}(T-t)}\left(\lambda^{\mathbb{P}}(T-t)\right)^n}{n!} \mathcal{BS}\left(T-t, X^{(n)}\left(t\right), \sigma\right), \quad (5.20)$$

where

$$X^{(n)}\left(t\right) = X\left(t\right)\exp\left(nJ - \lambda^{\mathbb{P}}(T-t)\exp\left(J\right) + \lambda^{\mathbb{P}}(T-t)\right)$$

and $\mathcal{BS}$ denotes the Black-Scholes price of the corresponding European option. Note that the formula presented in Cont and Tankov [2004] involves the case of normally distributed jumps. We here face the special situation of a fixed jump size, so a normal distribution with mean J and volatility equal to zero. To be precise we here only calculate an approximation to (5.20) as it involves an infinte sum. Fortunately this sum decreases very rapidly and we stop when reaching the first summand less than 0.01.

For step 1b we again choose for $t_j \leq t \leq t_{j+1}$ the set of

$$\left\{1,\ \mathrm{Pol}_2\left(X\left(t\right)\right),\ \mathrm{Pol}_3\left(E_\Pi\left(t, X\left(t\right), t_{j+1}\right)\right),\ \mathrm{Pol}_3\left(E_\Pi\left(t, X\left(t\right), t_L\right)\right)\right\}$$

for m_t^M and ϕ_t^M. The basis functions for the Brownian motion driven part of the BSDE, namely ψ_t^M and for the jump part, namely $\tilde{\psi}_t^M$, are both given by the set

$$\left\{1,\ X\left(t\right)\frac{\partial E_\Pi\left(t, X\left(t\right), t_{j+1}\right)}{\partial X\left(t\right)},\ X\left(t\right)\frac{\partial E_\Pi\left(t, X\left(t\right), t_L\right)}{\partial X\left(t\right)}\right\}.$$

Due to the presence of jumps the results are based on 5000 trajectories for all steps of the algorithm. For step 2 again, ψ_t^M and $\tilde{\psi}_t^M$ are enlarged by the martingale and maximum process as given in the process $\mathcal{X}$.

Standard expectation

In Table 5.7 we calculate the standard expectation for different $\lambda^{\mathbb{P}}$.

x_0	$\lambda^{\mathbb{P}} = 0,\ J = 0$	$\lambda^{\mathbb{P}} = 1,\ J = 0.06$	$\lambda^{\mathbb{P}} = 3,\ J = 006.$
90	11.8341	12.0982	12.8914
	(11.7254)	(11.9410)	(12.6078)
100	6.4077	6.7406	7.4556
	(6.3421)	(6.6429)	(7.3332)
110	3.2178	3.5036	4.0981
	(3.1655)	(3.4419)	(4.0025)

Table 5.7.: standard expectation - jump diffusion case

Kullback-Leibler divergence

Following in the Tables 5.8, 5.9 and 5.10 we deal with Kullback-Leibler divergence and present results for different α and $\lambda^{\mathbb{P}}$.

x_0	$\alpha = 5$	$\alpha = 10$	$\alpha = 100$	$\alpha = 10^4$	$\alpha = 10^6$
90	10.1392	10.4167	11.5438	11.8309	11.8341
	(10.0000)	(10.0000)	(11.4024)	(11.7219)	(11.7253)
100	3.7198	4.6258	6.1446	6.4049	6.4077
	(3.4854)	(4.4943)	(6.0716)	(6.3392)	(6.3421)
110	1.7037	2.1946	3.0640	3.2162	3.2178
	(1.5749)	(2.1043)	(3.0105)	(3.1640)	(3.1655)

Table 5.8.: Kullback-Leibler divergence driver for different α with $\lambda^{\mathbb{P}} = 0,\ J = 0$ - jump diffusion case

x_0	$\alpha = 5$	$\alpha = 10$	$\alpha = 100$	$\alpha = 10^4$	$\alpha = 10^6$
90	10.1742	10.5406	11.7879	12.0948	12.0981
	(10.0000)	(10.0000)	(11.6113)	(11.9642)	(11.9684)
100	3.8556	4.8457	6.4593	6.7376	6.7406
	(3.5358)	(4.6205)	(6.3598)	(6.6549)	(6.6581)
110	1.8160	2.3676	3.3333	3.5018	3.5036
	(1.6534)	(2.2464)	(3.2575)	(3.4319)	(3.4338)

Table 5.9.: Kullback-Leibler divergence driver for different α with $\lambda^{\mathbb{P}} = 1,\ J = 0.06$ - jump diffusion case

As can be seen from Table 5.7 the case of $\alpha = 10^6$ nearly gives the results of the case of standard expectation.

x_0	$\alpha = 5$	$\alpha = 10$	$\alpha = 100$	$\alpha = 10^4$	$\alpha = 10^6$
90	10.3872	11.0005	12.5179	12.8873	12.8913
	(10.0000)	(10.0000)	(12.0485)	(12.4676)	(12.4714)
100	4.1867	5.2749	7.1186	7.4520	7.4555
	(3.6574)	(4.9070)	(6.8778)	(7.2078)	(7.2112)
110	2.1074	2.7411	3.8892	4.0959	4.0981
	(1.8185)	(2.5292)	(3.7244)	(3.9358)	(3.9381)

Table 5.10.: Kullback-Leibler divergence driver for different α with $\lambda^{\mathbb{P}} = 3$, $J = 0.06$ - jump diffusion case

Worst case with mean partially known

The parameters involved are given as follows: $B^+ = 0.5$, $B^- = -0.5$, $d^+ = 0.5$, $d^- = -0.25$. The Tables are organized similar to the previous times. We alternate fixing either $\mu^+ = \mu$ or $\mu^- = \mu$ while looking at different jump intensities $\lambda^{\mathbb{P}}$.

x_0	$\mu^+ = 0.04$, $\mu^- = -0.96$	$\mu^+ = 0.04$, $\mu^- = -0.06$	$\mu^+ = 0.04$, $\mu^- = 0.04$
90	11.8344	11.8344	11.8341
	(11.7248)	(11.7248)	(11.7254)
100	6.4078	6.4078	6.4077
	(6.3413)	(6.3413)	(6.3421)
110	3.2180	3.2180	3.2178
	(3.1649)	(3.1649)	(3.1655)

Table 5.11.: worst case with mean partially known driver for $\lambda^{\mathbb{P}} = 0$, $J = 0$, $\mu^+ = 0.04$ and different choices of μ^- - jump diffusion case

x_0	$\mu^+ = 1.04$, $\mu^- = 0.04$	$\mu^+ = 0.14$, $\mu^- = 0.04$	$\mu^+ = 0.04$, $\mu^- = 0.04$
90	10.4826	10.4826	11.8341
	(10.0000)	(10.0000)	(11.7254)
100	4.2627	4.2627	6.4077
	(4.1248)	(4.1248)	(6.3421)
110	1.7366	1.7366	3.2178
	(1.6182)	(1.6182)	(3.1655)

Table 5.12.: worst case with mean partially known driver for $\lambda^{\mathbb{P}} = 0$, $J = 0$, $\mu^- = 0.04$ and different choices of μ^+ - jump diffusion case

x_0	$\mu^+ = 0.04$, $\mu^- = -0.96$	$\mu^+ = 0.04$, $\mu^- = -0.06$	$\mu^+ = 0.04$, $\mu^- = 0.04$
90	12.0479 (11.9019)	12.0478 (11.9021)	12.0471 (11.9042)
100	6.6762 (6.5773)	6.6758 (6.5774)	6.6735 (6.5810)
110	3.4464 (3.3654)	3.4461 (3.3657)	3.4441 (3.3683)

Table 5.13.: worst case with mean partially known driver for $\lambda^{\mathbb{P}} = 1$, $J = 0.06$, $\mu^+ = 0.04$ and different choices of μ^- - jump diffusion case

x_0	$\mu^+ = 1.04$, $\mu^- = 0.04$	$\mu^+ = 0.14$, $\mu^- = 0.04$	$\mu^+ = 0.04$, $\mu^- = 0.04$
90	10.4830 (10.0000)	10.6202 (10.0000)	12.0471 (11.9042)
100	4.2165 (3.9591)	4.5656 (4.3366)	6.6735 (6.5810)
110	1.7220 (1.5491)	1.9518 (1.7943)	3.4441 (3.3683)

Table 5.14.: worst case with mean partially known driver for $\lambda^{\mathbb{P}} = 1$, $J = 0.06$, $\mu^- = 0.04$ and different choices of μ^+ - jump diffusion case

x_0	$\mu^+ = 0.04$, $\mu^- = -0.96$	$\mu^+ = 0.04$, $\mu^- = -0.06$	$\mu^+ = 0.04$, $\mu^- = 0.04$
90	12.8470 (12.4069)	12.8468 (12.4071)	12.8458 (12.4091)
100	7.3986 (7.1357)	7.3981 (7.1357)	7.3954 (7.1403)
110	4.0460 (3.8720)	4.0456 (3.8727)	4.0428 (3.8780)

Table 5.15.: worst case with mean partially known driver for $\lambda^{\mathbb{P}} = 3$, $J = 0.06$, $\mu^+ = 0.04$ and different choices of μ^- - jump diffusion case

x_0	$\mu^+ = 1.04$, $\mu^- = 0.04$	$\mu^+ = 0.14$, $\mu^- = 0.04$	$\mu^+ = 0.04$, $\mu^- = 0.04$
90	10.9016 (10.0000)	11.1047 (10.1912)	12.8458 (12.4091)
100	4.7657 (4.3865)	5.1576 (4.7997)	7.3954 (7.1403)
110	2.1398 (1.8677)	2.4110 (2.1533)	4.0428 (3.8780)

Table 5.16.: worst case with mean partially known driver for $\lambda^{\mathbb{P}} = 3$, $J = 0.06$, $\mu^- = 0.04$ and different choices of μ^+ - jump diffusion case

Good-deal bounds

Due to the presence of jumps we here consider good-deal bounds for different Λ and $\lambda^{\mathbb{P}}$.

x_0	$\lambda^{\mathbb{P}} = 1$, $\Lambda = 0.5$	$\lambda^{\mathbb{P}} = 3$, $\Lambda = 0.5$	$\lambda^{\mathbb{P}} = 1$, $\Lambda = 1$	$\lambda^{\mathbb{P}} = 3$, $\Lambda = 1$
90	11.4997 (11.2707)	12.1071 (11.6837)	10.4439 (10.0000)	10.7011 (10.0000)
100	6.0565 (5.9043)	6.5356 (6.3561)	4.0455 (3.7779)	4.2038 (3.9169)
110	2.9763 (2.8715)	3.2928 (3.1472)	1.5277 (1.3471)	1.4672 (1.2205)

Table 5.17.: good-deal bounds driver for $J = 0.06$ and different Λ - jump diffusion case

Worst case with ball scenarios

In the worst case with ball scenarios situation we state results for different constellations of δ_1 and δ_2. These are given in Table 5.18 for $\lambda^{\mathbb{P}} = 1$ and in Table 5.19 for $\lambda^{\mathbb{P}} = 3$.

x_0	$\delta_1 = 0.5,\ \delta_2 = 0.5$	$\delta_1 = 0.5,\ \delta_2 = 1$	$\delta_1 = 1,\ \delta_2 = 0.5$
90	10.4790 (10.0000)	10.3741 (10.0000)	10.1983 (10.0000)
100	4.2241 (3.9417)	3.9182 (3.6057)	3.1273 (2.6565)
110	1.7314 (1.5357)	1.5393 (1.3234)	1.0532 (0.7634)

Table 5.18.: worst case with ball scenarios driver for $\lambda^{\mathbb{P}} = 1$ and different δ_1, δ_2 - jump diffusion case

The calculation of a specific result in the jump diffusion setting takes less then 2 minutes.

x_0	$\delta_1 = 0.5,\ \delta_2 = 0.5$	$\delta_1 = 0.5,\ \delta_2 = 1$	$\delta_1 = 1,\ \delta_2 = 0.5$
90	10.9015	10.7421	10.4288
	(10.0000)	(10.0000)	(10.0000)
100	4.7729	4.4381	3.582
	(4.3702)	(4.0012)	(3.0189)
110	2.1471	1.9184	1.3416
	(1.8559)	(1.6077)	(0.9623)

Table 5.19.: worst case with ball scenarios driver for $\lambda^{\mathbb{P}} = 3$ and different δ_1, δ_2 - jump diffusion case

5.4.3. Optimal entrance problem

We here consider the optimal entrance problem in a geometric Brownian motion setting, meaning $J = \lambda^{\mathbb{P}} = 0$. Let us define the grid of exercising dates by $t_j = j\Delta^c$, $j = 0, ..., T/\Delta^c$. Here $1/\Delta^c$ gives the number of exercising dates over a year. For the fine grid we work with $\Delta = \Delta^c/10$. The payoff is given by

$$\Pi\left(t, X\left(t\right)\right) = K \exp\left(-rt\right)$$

for a fixed irreversible cost K and

$$h\left(t, X\left(t\right)\right) = \left(X\left(t\right) - c\right) \exp\left(-rt\right)$$

which measure the payoff after entering the market minus the running costs c, taking into account discounting. We choose for step 1a and 1b

$$\left\{1,\ \mathrm{Pol}_3\left(X\left(t\right)\right),\ \mathrm{Pol}_3\left(h\left(t, X\left(t\right)\right)\right)\right\}$$

as the set of basis functions for m_t^M. The basis functions for the Brownian motion driven part of the BSDE are given by the set

$$\left\{1,\ X\left(t\right) \frac{\partial h\left(t, X\left(t\right)\right)}{\partial X\left(t\right)}\right\}.$$

In step 2 of our algorithm we add the martingale and the maximum process to the set of basis functions of the previous sets as usual. We are going to use 5000 trajectories in every step of our algorithm.

We do not consider lower bounds here. As can be found in the paper of Dixit [1989] it is not optimal to exercise as soon as the project has a positive expected output. So using this suboptimal stopping times would lead to lower values. We therefore left this as an open question for future research.

Standard expectation

First we consider the case of standard expectation working under the following setting with respect to the reference measure $\mathbb{P}$:

$$r = 0.1,\ \delta = 0.1,\ \sigma = 0.1,\ c = 1,\ K = 1,\ T = 100 \text{ yrs.}$$

In Table 5.20 we calculate the problem for different values of Δ^c and x_0 while multiplying the payoff h with Δ^c. This normalization is in fact a discrete approximation to the continuous payoff stream after entering the project which is given by an integral. In that sense Table 5.20 can be seen as a rough approximation to the continuous time problem with infinite time horizon as considered in Dixit [1989] for example. Here the last column gives the corresponding value by Dixit [1989].

x_0	$1/\Delta^c = 1$	$1/\Delta^c = 10$	$1/\Delta^c = \infty$
1	0.7843	0.7662	0.5595
1.375	3.2188	3.0087	2.7500
1.5	4.4733	4.2579	4.0000

Table 5.20.: standard expectation case for different Δ^c and initial values x_0 - optimal entrance case

The initial value 1.375 would be the entrance boundary given by Dixit [1989] for the parameter setting considered here.

Kullback-Leibler

For the rest of this Section we basically follow the ideas of Section 5.4.1 considering the Kullback-Leibler divergence driver for different α where the values of the last column follow from Table 5.20, so the standard expectation case.

Worst case with mean partially known

Again we consider the worst case with mean partially known where the values in Table 5.22 are mainly determined by the fact that we fix $\mu^- = \mu$. In Table 5.23 we fix $\mu^+ = \mu$ and slightly change the parameters. Remember that μ is given by $r - \delta = 0$. This is done cause the option got worthless very easily due to the long time horizon. Furthermore

x_0	$\alpha = 5$	$\alpha = 10$	$\alpha = 100$	$\alpha = 10^4$	$\alpha = 10^6$	$\alpha = \infty$
1	0.2783	0.4015	0.7012	0.7655	0.7662	0.7662
1.375	0.6910	1.1405	2.6403	3.0047	3.0087	3.0087
1.5	0.8861	1.5732	3.8327	4.2532	4.2578	4.2579

Table 5.21.: Kullback-Leibler divergence driver for different α - optimal entrance case

x_0	$\mu^+ = 1$, $\mu^- = 0$	$\mu^+ = 0.1$, $\mu^- = 0$	$\mu^+ = 0.05$, $\mu^- = 0$	$\mu^+ = 0$, $\mu^- = 0$
1	0.7696	0.71382	0.7354	0.7662
1.375	2.9677	2.9763	2.9875	3.0087
1.5	4.2204	4.2361	4.2443	4.2579

Table 5.22.: worst case with mean partially known driver for $\mu^- = 0$ and different choices of μ^+ - optimal entrance case

x_0	$\mu^+ = 0$, $\mu^- = -0.05$	$\mu^+ = 0$, $\mu^- = -0.03$	$\mu^+ = 0$, $\mu^- = -0.01$	$\mu^+ = 0$, $\mu^- = 0$
1	0.0677	0.1758	0.4668	0.7662
1.375	0.4862	0.8810	2.0261	3.0087
1.5	0.7041	1.4132	3.1503	4.2579

Table 5.23.: worst case with mean partially known driver for $\mu^+ = 0$ and different choices of μ^- - optimal entrance case

$B^+ = 1000$ and $B^+ = -1000$ are chosen such that the driver is practically independent of these parameters.

As explained in Section 5.4.1, the worst case with mean partially known driver equals the worst case with ball scenarios driver for certain parameters due to the absence of jumps. The calculations took approximately an hour.

5.5. Proofs

Proof. of (5.1)-(5.2) and Proposition 57: The proof uses results obtained by Krätschmer and Schoenmakers [2010] in a discrete-time setting with $h = 0$. By time-consistency of U, a property that is preserved with respect to stopping times, i.e., for any stopping time τ with $0 \leq t \leq \tau \leq T$ (by backward induction),

$$U(t) = U(t) \circ U(\tau),$$

we have

$$\sup_{\tau \in \mathcal{T}} U(0, H(\tau)) = \sup_{\tau \in \mathcal{T}} U(0, U(\tau, H(\tau))) = \sup_{\tau \in \mathcal{T}} U(0, \tilde{H}(\tau)),$$

where $\tilde{H}(t) := U(t, H(t))$ for $t \in [0, T]$. Hence, the optimal stopping problem (5.4) with non-adapted rewards $(H(t))_{t \in \mathcal{T}}$ can be transformed into an (equivalent) optimal stopping problem with adapted rewards $(\tilde{H}(t))_{t \in \mathcal{T}}$. Therefore, the existence of an optimal stopping time in (5.1) follows, upon continuous embedding, as a consequence of Theorem 3.2 in Krätschmer and Schoenmakers [2010]. Furthermore, upon continuous embedding, (5.2) follows as a consequence of Theorem 3.4 in Krätschmer and Schoenmakers [2010] and Proposition 57 is a consequence of Theorem 5.4 in the same reference. ■

Proof. of Theorem 59: For a square-integrable H that is $\mathcal{F}_T$-adapted and $t \in [0,T]$, let us consider

$$U^h(t) = \inf_{(q,\lambda)\in C}\left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[H + \sum_{t\le t_j} h(t_j, X(t_j))\right] + c(t,\mathbb{Q})\right\}, \tag{5.21}$$

where (for the first part of the proof) $H = 0$. Of course, $U^h(t_j) = U^h(t_j+) + h(t_j, X(t_j))$ and by time-consistency, for $t \in (t_j, t_{j+1}]$,

$$U^h(t) = \inf_{(q,\lambda)\in C}\left\{\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[U^h(t_{j+1})\right] + c(t,\mathbb{Q})\right\}. \tag{5.22}$$

The first part of (a) would follow if we could show that there exist predictable, square-integrable $(Z, \tilde{Z})$ such that

$$dU^h(t) = -g\left(t, Z(t), \tilde{Z}(t)\right)dt + Z(t)\,dW(t) + \tilde{Z}(t)\,d\tilde{N}(t), \quad \text{for } t_j < t \le t_{j+1}, \tag{5.23}$$

with $j = 0, \ldots, L-1$. Let $t_j < t \le t_{j+1}$. Notice that an adapted process, say Y, satisfying the RHS of (5.23) may be seen as a solution to a BSDE. To be more precise, by Tang and Li [1994], there exists a unique triple of processes, say $(Y(t), Z(t), \tilde{Z}(t))_{t_j\le t\le t_{j+1}} \in \mathcal{S}^2 \times L^2(d\mathbb{P}\times ds) \times L^2(d\mathbb{P}\times ds)$, satisfying

$$dY(t) = -g\left(t, Z(t), \tilde{Z}(t)\right)dt + Z(t)\,dW(t) + \tilde{Z}(t)\,d\tilde{N}(t), \quad \text{and} \quad Y(t_{j+1}) = U^h(t_{j+1}),$$

where we denote by $\mathcal{S}^2$ the space of all processes for which the maximum is square-integrable. We need to show that $U^h(t) = Y(t)$ for $t_j < t \le t_{j+1}$. Let $Q \in \mathcal{Q}$. We write

$$\begin{aligned}
Y(t) &= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}[Y(t)] \\
&= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[U^h(t_{j+1}) + \int_t^{t_{j+1}} g\left(s, Z(s), \tilde{Z}(s)\right) ds - \int_t^{t_{j+1}} Z(s)\, dW(s) - \int_t^{t_{j+1}} \tilde{Z}(s)\, d\tilde{N}(s)\right] \\
&= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[U^h(t_{j+1}) + \int_t^{t_{j+1}} \left[-q(s) Z(s) - \tilde{Z}(s)\left(\lambda(s) - \lambda^{\mathbb{P}}\right) + g\left(s, Z(s), \tilde{Z}(s)\right)\right] ds \right.\\
&\qquad \left. + \int_t^{t_{j+1}} Z(s)\, dW^{\mathbb{Q}}(s) + \int_t^{t_{j+1}} \tilde{Z}(s)\, d\tilde{N}^{\mathbb{Q}}(s)\right] \\
&= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[U^h(t_{j+1}) + \int_t^{t_{j+1}} \left[-q(s) Z(s) - \tilde{Z}(s)\left(\lambda(s) - \lambda^{\mathbb{P}}\right) + g\left(s, Z(s), \tilde{Z}(s)\right)\right] ds\right] \\
&\leq \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\left[U^h(t_{j+1}) + \int_t^{t_{j+1}} r\left(s, q(s), \lambda(s) - \lambda^{\mathbb{P}}\right) ds\right], \qquad (5.24)
\end{aligned}$$

where we used in the first equality that $Y(t)$ is $\mathcal{F}_t$-measurable. Note that the conditional expectation in the first equality is well-defined by the inequality of Cauchy-Schwarz, as (q, λ) take values in a compact set and Y is square-integrable under $\mathbb{P}$. The third and fourth equalities hold because $\int_{t_j}^{t} Z(s)\, dW^{\mathbb{Q}}(s)$ and $\int_{t_j}^{t} \tilde{Z}(s)\, \tilde{N}^{\mathbb{Q}}(s)$ are well-defined martingales, since for any $\mathbb{Q}$ with (q, λ) in a compact bounded set we have, again by Cauchy-Schwarz,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\sqrt{\int_{t_j}^{t_{j+1}} \|Z(s)\|^2 ds}\right] &= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\sqrt{\int_{t_j}^{t_{j+1}} \|Z(s)\|^2 ds}\right] \\
&\leq \sqrt{\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2\right]}\sqrt{\mathbb{E}\left[\int_{t_j}^{t_{j+1}} \|Z(s)\|^2 ds\right]} < \infty,
\end{aligned}$$

and a similar argument holds for $\tilde{Z}$. It follows from (5.24) and the fact that we can restrict the infimum in (5.22) to $\mathbb{Q} \in C$ that

$$Y(t) \leq U^h(t).$$

Next, by a measurable selection theorem (see e.g., Beněs [1970]), choose predictable $(q(s), \lambda(s) - \lambda^{\mathbb{P}}) \in \partial g(s, Z(s), \tilde{Z}(s))$. Then, q and λ induce an equivalent probability measure, $\mathbb{Q}^g$, with Radon-Nikodym derivative given by (5.5). Proceeding as in (5.24) with q, λ and $\mathbb{Q}^g$ (where the inequality in (5.24) becomes an equality) yields

$$Y(t) = \mathbb{E}_{\mathbb{Q}^g}^{\mathcal{F}_t}\left[U^h(t_{j+1}) + \int_t^{t_{j+1}} r\left(s, q(s), \lambda(s) - \lambda^{\mathbb{P}}\right) ds\right]. \tag{5.25}$$

Thus, by the definition of $U^h(t)$ in (5.22), we get $Y(t) \geq U^h(t)$. Therefore, indeed $Y(t) = U^h(t)$ for all $t_j < t \leq t_{j+1}$. This shows (5.9). (5.10) is seen similarly by setting $h = 0$ in (5.21). This proves part (a) of the theorem.

To see part (b), note that by part (a), there exist square-integrable $(Z^*, \tilde{Z}^*)$ such that (5.11) holds. Hence,

$$\begin{aligned}
V^*(t_{j+1}) - U(t_j, V^*(t_{j+1})) &= M^{*,g}(t_{j+1}) - A^{*,g}(t_{j+1}) - U(t_j, M^{*,g}(t_{j+1}) \\
&\quad + A^{*,g}(t_{j+1})) \\
&= M^{*,g}(t_{j+1}) - M^{*,g}(t_j) \\
&= \int_{t_j}^{t_{j+1}} Z^*(s)\, dW(s) + \int_{t_j}^{t_{j+1}} \tilde{Z}^*(s)\, d\tilde{N}(s) \\
&\quad - \int_{t_j}^{t_{j+1}} g(s, Z^*(s), \tilde{Z}^*(s)) ds.
\end{aligned}$$

From (5.4), part (b) follows. ■

Proof. of (5.15): We now show that our approximation scheme converges. Suppose that equations (5.1)-(5.7) hold with a square-integrable p-dimensional Markov process, $\mathcal{X}$, and an arbitrary function (driver) $g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ that is uniformly Lipschitz continuous in $(z, \tilde{z})$. The following theorem establishes convergence of our approximation scheme:

Theorem 68 *We have that*

$$\begin{aligned}
&\lim_{\pi \to 0} \lim_{M \to \infty} \lim_{N \to \infty} Y^{\pi,M,N}(T_0) \to Y(T_0) \ \ in\ L^2, \\
&\lim_{\pi \to 0} \lim_{M \to \infty} \lim_{N \to \infty} Z^{\pi,M,N} \to Z \ \ in\ L^2(dP \times ds, \Omega \times [0,T]), \\
&\lim_{\pi \to 0} \lim_{M \to \infty} \lim_{N \to \infty} \tilde{Z}^{\pi,M,N} \to \tilde{Z} \ \ in\ L^2(dP \times ds, \Omega \times [0,T]).
\end{aligned}$$

Proof. It follows from Bouchard and Elie [2008] that $Y^{\pi}(t)$ converges to $Y(t)$ in L^2. From this and Lemma 69 below we may conclude that it is sufficient to prove that $Y^{\pi,M,N}(T_0)$ converges to $Y^{\pi,M}(T_0)$ in L^2, which would follow if

$$\lim_{N \to \infty} Y^{\pi,M,N}(T_0) \to Y^{\pi,M}(T_0) \ \text{ in } L^2.$$

And this follows from Lemma 70 below. The proof for $Z^{\pi,M,N}$ and $\tilde{Z}^{\pi,M,N}$ is similar. ■

Lemma 69 *For every $t \in [T_0, T_1]$ and for fixed π, we have that $Y^{\pi,M}(t) \to Y^{\pi}(t)$, $Z^{\pi,M}(t) \to Z^{\pi}(t)$ and $\tilde{Z}^{\pi,M}(t) \to \tilde{Z}^{\pi}(t)$ in L^2 as M tends to infinity.*

Proof. The lemma would follow if we could show by a backward induction that, for every s_{jp}, we have $Y^{\pi,M}(s_{jp}) \to Y^{\pi}(s_{jp})$, $Z^{\pi,M}(s_{jp}) \to Z^{\pi}(s_{jp})$ and $\tilde{Z}^{\pi,M}(s_{jp}) \to \tilde{Z}^{\pi}(s_{jp})$ in L_1^2, L_d^2 and L_k^2, respectively. Since our basis functions span the entire space, $L_1^2(\mathcal{F}_{s_{jp}})$, the lemma clearly holds for $s_{jp} = T$. (Without loss of generality we may set $Z^{\pi,M}(T_1) = Z^{\pi}(T_1)$ and $\tilde{Z}^{\pi,M}(T_1) = \tilde{Z}^{\pi}(T_1)$.) It will be useful to consider the projection onto the span of $\psi^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp}))$, and $\tilde{\psi}^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp}))$, say $\hat{P}^{\pi,M}$ and $\hat{\tilde{P}}^{\pi,M}$, respectively, instead of the projection onto the span of $\psi^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp}))\Delta W_{jp}$ and $\tilde{\psi}^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp}))\Delta\bar{\tilde{N}}_{jp}$, respectively. We write

$$\begin{aligned}
\gamma_{s_{jp}}^{\pi,M}\psi^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp})) &= \hat{P}^{\pi,M}\left(Y^M\left(s_{j(p+1)}\right)\Delta W_{jp}\right)/\mathbb{E}\left[\Delta^2 W_{jp}\right] \\
&= \hat{P}^{\pi,M}\Big(\mathbb{E}^{s_{jp}}\left[Y_{s_{j(p+1)}}^{\pi,M}\Delta W_{jp}\right]\Big)/\mathbb{E}\left[\Delta^2 W_{jp}\right] \\
&\overset{M\to\infty}{\to} \mathbb{E}^{s_{jp}}\left[Y^{\pi}\left(s_{j(p+1)}\right)\Delta W_{jp}\right]/\mathbb{E}\left[\Delta^2 W_{jp}\right] = Z^{\pi}(s_{jp}),
\end{aligned}$$

in L^2, where we used (5.2)-(5.4) in the first equation. Furthermore,

$$\begin{aligned}
\tilde{\gamma}_{s_{jp}}^{\pi,M}\tilde{\psi}^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp})) &= \hat{\tilde{P}}^{\pi,M}\left(Y^M\left(s_{j(p+1)}\right)\Delta\bar{\tilde{N}}_{jp}\right)/\mathbb{E}\left[\Delta^2\bar{\tilde{N}}_{jp}\right] \\
&= \hat{\tilde{P}}^{\pi,M}\Big(\mathbb{E}^{s_{jp}}\left[Y^{\pi,M}\left(s_{j(p+1)}\right)\Delta\tilde{N}_{jp}\right]\Big)/\mathbb{E}\left[\Delta^2\tilde{N}_{jp}\right] \\
&\overset{M\to\infty}{\to} \mathbb{E}^{s_{jp}}\left[Y^{\pi}\left(s_{j(p+1)}\right)\Delta\tilde{N}_{jp}\right]/\mathbb{E}\left[\Delta^2\tilde{N}_{jp}\right] = \tilde{Z}^{\pi}(s_{jp}),
\end{aligned}$$

in L^2, where we used (5.3)-(5.5) in the first equation. The convergence then follows since, by the induction assumption, we have that $\mathbb{E}^i\left[Y^{\pi,M}\left(s_{j(p+1)}\right)\Delta W_{jp}\right]$ converges in L^2 to $\mathbb{E}_i\left[Y^{\pi}\left(s_{j(p+1)}\right)\Delta W_{jp}\right]$ as M tends to infinity. The lemma is now a consequence of (5.1) and (5.6). ■

Lemma 70 *For all j, we have that $\alpha_{s_{jp}}^{\pi,M,N} \to \alpha_{s_{jp}}^{\pi,M}$, $\gamma_{s_{jp}}^{\pi,M,N} \to \gamma_{s_{jp}}^{\pi,M}$ and $\tilde{\gamma}_{s_{jp}}^{\pi,M,N} \to \tilde{\gamma}_{s_{jp}}^{\pi,M}$ as N tends to infinity.*

Proof. By the Law of Large Numbers (LLN), we have that $(A_{s_{jp}}^{\pi,M,N})$, $(\bar{A}_{s_{jp}}^{\pi,M,N})$ and $(\tilde{A}_{s_{jp}}^{\pi,M,N})$ converge to $(A_{s_{jp}}^{\pi,M})$, $(\bar{A}_{s_{jp}}^{\pi,M})$ and $(\tilde{A}_{s_{jp}}^{\pi,M})$, respectively. We prove the claim by a backward induction. For $\alpha, \gamma, \tilde{\gamma} \in \mathbb{R}^M$ and $x \in \mathbb{R}^d$ set

$$\begin{aligned}
&F(T_1, \alpha, \gamma, \tilde{\gamma}, x) := w(x) \\
&F(s_{jp}, \alpha, \gamma, \tilde{\gamma}, x) := \alpha m^M(s_{jp}, x) + g(s_{jp}, \gamma\psi^M(s_{jp}, x), \tilde{\gamma}\tilde{\psi}^M(s_{jp}, x))\Delta_{jp} \quad \text{for } s_{jp} < T_1.
\end{aligned}$$

Furthermore, for every j, p, $F(s_{jp}, \cdot)$ is continuous in x and Lipschitz continuous in $(\alpha, \gamma, \tilde{\gamma})$. Moreover, by the LLN we have that

$$\frac{1}{N}\sum_{n=1}^{N} F\left(s_{j(p+1)}, \alpha^{\pi,M}_{s_{j(p+1)}}, \gamma^{\pi,M}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M}_{s_{j(p+1)}}, \mathcal{X}^{\pi,n}\left(s_{j(p+1)}\right)\right) m^M\left(s_{j(p+1)}, \mathcal{X}^{\pi,n}\left(s_{j(p+1)}\right)\right)$$
$$\overset{N\to\infty}{\to} \mathbb{E}\left[F\left(s_{j(p+1)}, \alpha^{\pi,M}_{s_{j(p+1)}}, \gamma^{\pi,M}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M}_{s_{j(p+1)}}, \mathcal{X}^{\pi}\left(s_{j(p+1)}\right)\right) m^M\left(s_{j(p+1)}, \mathcal{X}^{\pi}\left(s_{j(p+1)}\right)\right)\right].$$

Since, by Lipschitz continuity of g and the induction assumption, we have that

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}\left(F\left(s_{j(p+1)}, \alpha^{\pi,M,N}_{s_{j(p+1)}}, \gamma^{\pi,M,N}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M,N}_{s_{j(p+1)}}, \mathcal{X}^{\pi,n}\left(s_{j(p+1)}\right)\right)\right.\\
&\qquad \left. -F\left(s_{j(p+1)}, \alpha^{\pi,M}_{s_{j(p+1)}}, \gamma^{\pi,M}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M}_{s_{j(p+1)}}, \mathcal{X}^{\pi,n}\left(s_{j(p+1)}\right)\right)\right) m^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))\\
&\quad = \left(\left\|\alpha^{\pi,M,N}_{s_{j(p+1)}} - \alpha^{\pi,M}_{s_{j(p+1)}}\right\| + \left\|\gamma^{\pi,M,N}_{s_{j(p+1)}} - \gamma^{\pi,M}_{s_{j(p+1)}}\right\| + \left\|\tilde{\gamma}^{\pi,M,N}_{s_{j(p+1)}} - \tilde{\gamma}^{\pi,M}_{s_{j(p+1)}}\right\|\right)\\
&\qquad \times \frac{1}{N}\sum_{n=1}^{N} m^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp})) \overset{N\to\infty}{\to} 0,
\end{aligned}$$

it follows that

$$\frac{1}{N}\sum_{n=1}^{N} F\left(s_{j(p+1)}, \alpha^{\pi,M,N}_{s_{j(p+1)}}, \gamma^{\pi,M,N}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M,N}_{s_{j(p+1)}}, \mathcal{X}^{\pi,n}\left(s_{j(p+1)}\right)\right) m^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))$$
$$\overset{N\to\infty}{\to} \mathbb{E}\left[F\left(s_{j(p+1)}, \alpha^{\pi,M}_{s_{j(p+1)}}, \gamma^{\pi,M}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M}_{s_{j(p+1)}}, \mathcal{X}^{\pi}\left(s_{j(p+1)}\right)\right)\right] m^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp})).$$

Therefore,

$$\begin{aligned}
\alpha^{\pi,M,N}_{s_{jp}} &= \left(A^{\pi,M,N}_{s_{jp}}\right)^{-1} \frac{1}{N}\sum_{n=1}^{N} Y^{\pi,M,N}_{s_{j(p+1)}} m^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))\\
&= \left(A^{\pi,M,N}_{s_{jp}}\right)^{-1} \frac{1}{N}\sum_{n=1}^{N} F\left(s_{j(p+1)}, \alpha^{\pi,M,N}_{s_{j(p+1)}}, \gamma^{\pi,M,N}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M,N}_{s_{j(p+1)}},\right.\\
&\qquad \left.\mathcal{X}^{n,\pi}\left(s_{j(p+1)}\right)\right) m^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))\\
&\to \left(A^{\pi,M}_{s_{jp}}\right)^{-1} \mathbb{E}\left[F\left(s_{j(p+1)}, \alpha^{\pi,M}_{s_{j(p+1)}}, \gamma^{\pi,M}_{s_{j(p+1)}}, \tilde{\gamma}^{\pi,M}_{s_{j(p+1)}}\right.\right. ,\\
&\qquad \left.\left.\mathcal{X}^{\pi}\left(s_{j(p+1)}\right)\right) m^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp}))\right]\\
&= \left(A^{\pi,M}_{s_{jp}}\right)^{-1} \mathbb{E}\left[Y^{\pi,M}\left(s_{j(p+1)}\right) m^M(s_{jp}, \mathcal{X}^{\pi}(s_{jp}))\right] = \alpha^{\pi,M}_{s_{jp}}.
\end{aligned}$$

By replacing $\alpha^{\pi,M,N}_{s_{jp}}$ by $\gamma^{\pi,M,N}_{s_{jp}}$, $A^{\pi,M}_{j}$ by $\bar{A}^{\pi,M}_{s_{jp}}$, as well as $m^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))$ by $\psi^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))$, it follows similarly that $\gamma^{\pi,M,N}_{s_{jp}}$ converges to $\gamma^{\pi,M}_{s_{jp}}$. Also, by replacing $\alpha^{\pi,M,N}_{s_{jp}}$ by $\tilde{\gamma}^{\pi,M,N}_{s_{jp}}$, $\tilde{A}^{\pi,M}_{s_{jp}}$ by $\tilde{\bar{A}}^{\pi,M}_{s_{jp}}$, as well as $m^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))$ by $\tilde{\psi}^M(s_{jp}, \mathcal{X}^{\pi,n}(s_{jp}))$, it follows similarly that $\tilde{\gamma}^{\pi,M,N}_{s_{jp}}$ converges to $\tilde{\gamma}^{\pi,M}_{s_{jp}}$. This proves the induction. ■

Then, applying Theorem 16 above twice completes the proof of (5.15). ■

Proof. of Proposition 65: This follows from (5.25) in the proof of Theorem 59(a). ■

Proof. of Theorem 67: The stated convergence results follow as a consequence of our convergence results for BSΔEs (see the proof of (5.15)). Next, choose a fixed $n \in \{1, \ldots, N_3\}$. To show the biased high property, we then write

$$
\begin{aligned}
&\mathbb{E}\left[\widetilde{V}^{upp,N_3}(0)\right] \\
&= \mathbb{E}\Bigg[\frac{d\mathbb{Q}^{g,\pi,M,N_2,n}}{d\mathbb{P}}\mathbb{E}\Bigg[\max_{t_j\in\{0,\ldots,T\}}\Big(\Pi(t_j, X^n(t_j)) + U^{upper,h,\pi,M,N_3,n}(t_j) \\
&\qquad\qquad + M^{g,\pi,M,N_1,n}(T) - M^{g,\pi,M,N_1,n}(t_j)\Big) \\
&+ \sum_{j,p}\int_{s_{jp}}^{s_{j(p+1)}} r\left(s, q^{\pi,M,N_2,n}(s_{jp}), \lambda^{\pi,M,N_2,n}(s_{jp}) - \lambda^{\mathbb{P}}\right) ds \Big| X_1, \ldots, X_{N_4}\Bigg]\Bigg] \\
&\geq \mathbb{E}\Bigg[\frac{d\mathbb{Q}^{g,\pi,M,N_2,n}}{d\mathbb{P}}\Bigg[\max_{t_j\in\{0,\ldots,T\}}\Big(\Pi(t_j, X^n(t_j)) + u^h(t_j, X^n(t_j)) \\
&\qquad\qquad + M^{g,\pi,M,N_1,n}(T) - M^{g,\pi,M,N_1,n}(t_j)\Big) \\
&+ \sum_{j,p}\int_{s_{jp}}^{s_{j(p+1)}} r\left(s, q^{\pi,M,N_2,n}(s_{jp}), \lambda^{\pi,M,N_2,n}(s_{jp}) - \lambda^{\mathbb{P}}\right) ds\Bigg]\Bigg] \\
&\geq U\Big(0, \max_{t_j\in\{0,\ldots,T\}}\Big(\Pi(t_j, X^n(t_j)) + u^h(t_j, X^n(t_j)) + M^{g,\pi,M,N_1,n}(T) - M^{g,\pi,M,N_1,n}(t_j)\Big)\Big) \\
&\geq U\Big(0, \max_{t_j\in\{0,\ldots,T\}}\Big(\Pi(t_j, X^n(t_j)) + u^h(t_j, X^n(t_j)) + M^{g*,n}(T) - M^{g*,n}(t_j)\Big)\Big) = V^*(0),
\end{aligned}
$$

where we have used Proposition 66 and Jensen's inequality in the first inequality, (5.13) in the second inequality, and Proposition 57 in the last inequality and also in the last equality. ■

5.6. Conclusion and Outlook

We have developed a method to practically compute the solution to the optimal stopping problem in a general continuous-time setting featuring general time-consistent ambiguity averse preferences and general rewards driven by jump-diffusions. The resulting algorithm delivers an approximation to the solution that converges asymptotically to the true solution and yields a safe genuine (biased high) upper bound at the pre-limiting level. Additionally a lower bound is provided in the case of American options valuation using stopping times extracted from an approximation to the generalized continuation value taking into consideration ambiguity averse preferences. Our method is numerically efficient and eventually requires only simple least squares Monte Carlo regression techniques. Our method may be generalized to encompass multiple stopping problems, which we intend to consider in future research. Further the lower bounds for the optimal entrance case remain an open question as we would have to adopt to the non-trivial exercise boundaries found in Dixit [1989].

A. Appendix

In the Appendix we will present a short survey of the mathematical tools and concepts used in this work where we do not strive for completeness. For the interested reader we refer to Karatzas and Shreve [1991] and Øksendal [2003].

Let us for the Appendix assume that we are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with σ-algebra $\mathcal{F}$, filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ and probability measure $\mathbb{P}$, where $T < \infty$ is a finite time horizon. Let $\mathbb{F}$ satisfy the "usual conditions" as will be defined below.

A.1. Usual conditions

A filtration $\mathbb{F}$ satisfies the usual conditions if it

i) is completed by the $\mathbb{P}$-null sets,

ii) is right-continuous, $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{T \geq s > t} \mathcal{F}_s$, $\forall\ 0 \leq t \leq T$.

A.2. Martingale

Even it is well known, due to its importance for finance we shortly recall the notion of a martingale. Let $0 \leq r \leq s \leq T$ and $(X(t))_{0 \leq t \leq T}$ a process with $\mathbb{E}[\|X(t)\|] < \infty$ for all $t \in [0, T]$. Then X is said to be a submartingale (a supermartingale, respectively) if $\mathbb{E}^{\mathcal{F}_s}[X(t)] \geq X(s)$ ($\mathbb{E}^{\mathcal{F}_s}[X(t)] \leq X(s)$, respectively). X is a martingale if it is both, a sub- and a supermartingale.

Definition 71 *Let $(X(t))_{0 \leq t \leq T}$ be a stochastic process. If there exists a nondecreasing sequence $(\tau_n)_{n=1,\ldots,\infty}$ of $\mathbb{F}$-stopping times with $\mathbb{P}[\lim_{n \to \infty} \tau_n = \infty] = 1$ such that $(X(t \wedge \tau_n))_{0 \leq t \leq T}$ is a martingale for each $n \geq 1$, than X is said to be a local martingale.*

A.3. Martingale representation theorem

We will follow the presentation in Øksendal [2003] here. Therefore let $\mathcal{V} = V(A, B)$, $0 \leq A \leq B \leq T$ be the class of functions $f(t, \omega) : [0, T] \times \Omega \to \mathbb{R}$ with

i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, T]$

ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted

iii) $\mathbb{E}\left[\int_A^B f^2(t,\omega)\,dt\right] < \infty$

Theorem 72 *(Ito representation)Let X be a square-integrable random variable measurable with respect to $\mathbb{F}$. Then there exists a unique process $Z \in \mathcal{V}(0,T)$ such that*

$$X = \mathbb{E}[X] + \int_0^T Z^T(t)\,dW(t).$$

Further we have the martingale representation theorem.

Theorem 73 *Let $(X(t))_{0\le t\le T}$ be a square-integrable martingale with respect to $\mathbb{F}$. Then there exists a unique stochastic process $Z \in \mathcal{V}(0,T)$ such that*

$$X(t) = \mathbb{E}[X(0)] + \int_0^T Z^T(t)\,dW(t).$$

A.4. Girsanov theorem

The Girsanov theorem describes the dynamics of a stochastic process after the change of measure to an equivalent measure. Its has special importance in financial applications as we are able to change to the risk-neutral measure.

Theorem 74 *Let $X \in \mathcal{V}[0,T]$ (as defined in Appendix A.3). Define*

$$\mathcal{E}(X)(t) := \exp\left(\int_0^t X^T(s)\,dW(s) - \int_0^t \|X(s)\|^2\,ds\right), \quad t \le T \tag{A.1}$$

and assume that (A.1) is a martingale. Define the process $\widetilde{W}$ by

$$\widetilde{W}(t) := W(t) - \int_0^t X(s)\,ds, \quad t \le T.$$

Then $\widetilde{W}$ is a Brownian motion under the measure $\widetilde{\mathbb{P}}$ defined via $d\widetilde{\mathbb{P}} = \mathcal{E}(X)(T)\,d\mathbb{P}$.

A.5. Novikov condition

The Novikov condition ensures that the stochastic process (A.1) in Theorem 74 is a martingale.

Theorem 75 *Let $(X(t))_{0\le t\le T}$ be a measurable, adapted stochastic process satisfying*

$\mathbb{P}\left[\int_0^T X_i^2(t)\,dt < \infty\right] = 1$ *for all* i. *If*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|X(t)\|^2\,dt\right)\right] < \infty$$

we have $\mathcal{E}(X)(t)$ *as defined in (A.1) is a martingale.*

A.6. Ito rule-dynamics

Let $X = (X(t))_{0\leq t\leq T}$, $X \in \mathbb{R}^n$ be a continuous semimartingale. Given a function $f : [0,T]\times\mathbb{R}^n \to \mathbb{R} \in \mathcal{C}^{1,2}$ and $X(0)$ is $\mathcal{F}_0$-measurable the Ito formula holds.

$$\begin{aligned} f(t,X(t)) &= f(0,X(0)) + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} f(s,X(s))\,dX_i(s) \\ &\quad + \int_0^t \frac{\partial}{\partial s} f(s,X(s))\,ds + \frac{1}{2}\sum_{j,k=1}^n \int_0^t \frac{\partial^2}{\partial x_j \partial x_k} f(s,X(s))\,d\langle X_j, X_k\rangle(s) \end{aligned}$$

The product rule then follows by taking $f(t,x_1,x_2) = x_1 x_2$.

$$\begin{aligned} f(t,X_1(t),X_2(t)) &= f(0,X_1(0),X_2(0)) + \int_0^t X_2(s)\,dX_1(s) \\ &\quad + \int_0^t X_1(s)\,dX_2(s) + \frac{1}{2}\int_0^t d\langle X_1, X_2\rangle(s) \end{aligned} \tag{A.2}$$

A.7. Libor dynamics

Let W be a Brownian motion and consider the function

$$f(t,W(t)) = g(h(W(t)))\cdot i(j(W(t))) = g(X(t))\cdot i(Y(t))$$

with suitable regularity for g and i. By Ito's product rule (A.2) we have

$$\begin{aligned} d(g(X)\cdot i(Y)) &= \frac{\partial}{\partial X}(g(X)\cdot i(Y))\,dX + \frac{\partial}{\partial Y}(g(X)\cdot i(Y))\,dY \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial X^2}(g(X)\cdot i(Y))\,d\langle X\rangle + \frac{1}{2}\frac{\partial^2}{\partial Y^2}(g(X)\cdot i(Y))\,d\langle Y\rangle \\ &\quad + \frac{\partial^2}{\partial X \partial Y}(g(X)\cdot i(Y))\,d\langle X, Y\rangle\,. \end{aligned}$$

With $g(X) = X$ and $i(Y) = 1/Y$ it follows

$$d(X/Y) = \frac{1}{Y}dX - \frac{X}{Y^2}dY - \frac{X}{Y^3}d\langle Y\rangle - \frac{1}{Y^2}d\langle X, Y\rangle.$$

Taking the following dynamics $dX = \mu_X X dt + \sigma_X^T X dW$ and $dY = \mu_Y Y dt + \sigma_Y^T Y dW$ as they occur for example in the context of Libor modeling it follows

$$\begin{aligned}
d(X/Y) &= \frac{1}{Y}\left(\mu_X X dt + \sigma_X^T X dW\right) - \frac{X}{Y^2}\left(\mu_Y Y dt + \sigma_Y^T Y dW\right) \\
&\quad - \frac{X}{Y^3}(\sigma_Y Y)^2 dt - \frac{1}{Y^2}\sigma_X X \sigma_Y Y dt \\
&= \frac{1}{Y}\mu_X X dt - \frac{X}{Y}\sigma_Y^T \sigma_Y dt - \frac{1}{Y}\sigma_Y^T \sigma_X X dt - \frac{X}{Y}\mu_Y dt + \frac{1}{Y}\sigma_X^T X dW - \frac{X}{Y}\sigma_Y^T dW \\
&= \frac{X}{Y}\left(\mu_X - \mu_Y - \sigma_Y^T(\sigma_X - \sigma_Y)\right) dt + \frac{X}{Y}\left(\sigma_X^T - \sigma_Y^T\right) dW.
\end{aligned}$$

A.8. Swap rate dynamics

The derivation of the swap rate volatility (2.23) is essentially given in Schoenmakers [2005]. But in order to match to the present notation and to make reading more convenient, we now give a short recap. Let, exclusively in this Section, σ_r denote the volatility of the bond B_r, let μ_r be the drift of B_r, and λ be the market price of risk process with respect to the driving Brownian motion $\mathcal{W} = (W, \widehat{W}, \overline{W})$. That is, in the objective measure the zero bond dynamics are of the form

$$dB_r = \mu_r B_r dt + \sigma_r^T B_r d\mathcal{W}, \quad \text{with } \mu_r = \sigma_r^T \lambda$$

where $\sigma_{r,k} = 0$ for $m + \widehat{m} < k \le m + \widehat{m} + \overline{m}$. Following [Schoenmakers, 2005, p.17], we may write for $p \le r \le q$,

$$\begin{aligned}
dB_{p,q} &= \sum_{i=p}^{q-1} \delta_i \left(\mu_{i+1} B_{i+1} dt + \sigma_{i+1}^T B_{i+1} d\mathcal{W}\right) \\
&= \left(\sum_{i=p}^{q-1} \delta_i \mu_{i+1} B_{i+1}\right) dt + \left(\sum_{i=p}^{q-1} \delta_i \sigma_{i+1}^T B_{i+1}\right) d\mathcal{W} \\
&= \frac{B_{p,q}}{B_{p,q}} \left(\sum_{i=p}^{q-1} \delta_i \mu_{i+1} B_{i+1}\right) dt + \left(\sum_{i=p}^{q-1} \delta_i \sigma_{i+1}^T B_{i+1}\right) d\mathcal{W} \\
&= B_{p,q} \left(\sum_{i=p}^{q-1} \delta_i \mu_{i+1} \frac{B_{i+1}}{B_{p,q}}\right) dt + B_{p,q} \left(\sum_{i=p}^{q-1} \delta_i \sigma_{i+1}^T \frac{B_{i+1}}{B_{p,q}}\right) d\mathcal{W} \\
&= B_{p,q} \left(\sum_{i=p}^{q-1} w_i^{p,q} \mu_{i+1}\right) dt + B_{p,q} \left(\sum_{i=p}^{q-1} w_i^{p,q} \sigma_{i+1}^T\right) d\mathcal{W}.
\end{aligned}$$

with $w_i^{p,q} = \delta_i \frac{B_{i+1}}{B_{p,q}}$. We need

$$\begin{aligned}
d\left(B_r/B_{p,q}\right) = &\frac{B_r}{B_{p,q}}\left(\mu_r - \left(\sum_{i=p}^{q-1} w_i^{p,q}\mu_{i+1}\right)\right) \\
&- \left(\sum_{i=p}^{q-1} w_i^{p,q}\sigma_{i+1}^T\right)\left(\sigma_r - \left(\sum_{i=p}^{q-1} w_i^{p,q}\sigma_{i+1}\right)\right) dt \\
&+ \frac{B_r}{B_{p,q}}\left(\sigma_r^T - \left(\sum_{i=p}^{q-1} w_i^{p,q}\sigma_{i+1}^T\right)\right) dW \\
= &\frac{B_r}{B_{p,q}}\left(r + {\sigma_r}^T\theta - \left(\sum_{i=p}^{q-1} w_i^{p,q}\left(r + {\sigma_{i+1}}^T\theta\right)\right)\right) \\
&- \left(\sum_{i=p}^{q-1} w_i^{p,q}\sigma_{i+1}^T\right)\left(\sigma_r - \left(\sum_{i=p}^{q-1} w_i^{p,q}\sigma_{i+1}\right)\right) dt \\
&+ \frac{B_r}{B_{p,q}}\left(\sigma_r^T - \left(\sum_{i=p}^{q-1} w_i^{p,q}\sigma_{i+1}^T\right)\right) dW.
\end{aligned}$$

With $\sum_{i=p}^{q-1} w_i^{p,q} = \sum_{i=p}^{q-1} \delta_i \frac{B_{i+1}}{B_{p,q}} = 1$ we have

$$\frac{d(B_r/B_{p,q})}{B_r/B_{p,q}} = \left(\sigma_r^\top - \sum_{j=p}^{q-1} w_j^{p,q}\sigma_{j+1}^\top\right) d\mathcal{W}^{p,q},$$

so $B_r/B_{p,q}$ is a $\mathbb{P}_{p,q}$-martingale. Finally we obtain

$$\begin{aligned}
dS_{p,q} = d\frac{B_p - B_q}{B_{p,q}} = \\
\left[\frac{B_p}{B_{p,q}}\left(\sigma_p^\top - \sum_{j=p}^{q-1} w_j^{p,q}\sigma_{j+1}^\top\right) - \frac{B_q}{B_{p,q}}\left(\sigma_q^T - \sum_{j=p}^{q-1} w_j^{p,q}\sigma_{j+1}^\top\right)\right] d\mathcal{W}^{p,q} \\
= \left[\frac{B_p}{B_{p,q}}\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_p^\top - \sigma_{j+1}^\top\right) - \frac{B_q}{B_{p,q}}\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_q^\top - \sigma_{j+1}^\top\right)\right] d\mathcal{W}^{p,q} \\
= S_{p,q}\left[\frac{B_p}{B_p - B_q}\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_p^\top - \sigma_{j+1}^\top\right) - \frac{B_q}{B_p - B_q}\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_q^\top - \sigma_{j+1}^\top\right)\right] d\mathcal{W}^{p,q} \\
= S_{p,q}\left[\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_p^\top - \sigma_{j+1}^\top\right) + \frac{B_q}{B_p - B_q}\sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_p^\top - \sigma_q^\top\right)\right] d\mathcal{W}^{p,q} \\
=: S_{p,q}\Lambda_{p,q}^\top d\mathcal{W}^{p,q}.
\end{aligned}$$

Similar to (1.13) in Schoenmakers [2005] we get

$$\begin{aligned}
\Lambda_{p,q} &= \sum_{j=p}^{q-1} w_j^{p,q}\left(\sigma_p - \sigma_{j+1}\right) + \frac{B_q}{B_p - B_q} w_j^{p,q}\left(\sigma_p - \sigma_q\right) \\
&= \sum_{j=p}^{q-1} w_j^{p,q} \sum_{r=p}^{j}\left(\sigma_r - \sigma_{r+1}\right) + \frac{B_q}{B_p - B_q}\sum_{r=p}^{q-1}\left(\sigma_r - \sigma_{r+1}\right) \\
&= \sum_{r=p}^{q-1}\left(\sigma_r - \sigma_{r+1}\right)\left(\sum_{j=r}^{q-1} w_j^{p,q} + \frac{B_q}{B_p - B_q}\right) \\
&= \sum_{r=p}^{q-1}\begin{bmatrix}\sqrt{v_r}\beta_r \\ \gamma_r\end{bmatrix}\frac{\delta_r\left(L_r + \alpha_r\right)}{1 + \delta_r L_r}\left(\sum_{j=r}^{q-1} w_j^{p,q} + \frac{B_q}{B_p - B_q}\right).
\end{aligned}$$

Further, by (1.27) from Schoenmakers [2005], it holds that

$$d\mathcal{W}^{(n)} = d\mathcal{W} + (\lambda - \sigma_n)dt, \quad \text{and} \quad d\mathcal{W}^{p,q} = \lambda dt - \sum_{l=p}^{q-1} w_l^{p,q}\sigma_{l+1}dt + d\mathcal{W}.$$

Therefore, we finally have

$$\begin{aligned}
d\mathcal{W}^{p,q} &= d\mathcal{W}^{(n)} + \sigma_n dt - \sum_{l=p}^{q-1} w_l^{p,q}\sigma_{l+1}dt \\
&= d\mathcal{W}^{(n)} + dt\sum_{l=p}^{q-1} w_l^{p,q}\left(\sigma_n - \sigma_{l+1}\right) \\
&= d\mathcal{W}^{(n)} + dt\sum_{l=p}^{q-1} w_l^{p,q}\sum_{k=l+1}^{n-1}\left(\sigma_{k+1} - \sigma_k\right) \\
&= d\mathcal{W}^{(n)} - dt\sum_{l=p}^{q-1} w_l^{p,q}\sum_{k=l+1}^{n-1}\frac{\delta_k\left(L_k + \alpha_k\right)}{1 + \delta_k L_k}\begin{bmatrix}\sqrt{v_k}\beta_k \\ \gamma_k\end{bmatrix}.
\end{aligned}$$

The LMM is retrieved by considering $\mathcal{W} = W$, $\alpha \equiv 0$, $\gamma \equiv 0$, $\beta \equiv 1$

A.9. Strong solutions to SDE´s

Let $A : [0,T] \times \mathbb{R}^k \to \mathbb{R}^k$, $B : [0,T] \times \mathbb{R}^k \to \mathbb{R}^{k\times m}$ be two mappings and consider the following SDE

$$dX\left(t\right) = A\left(t, X\left(t\right)\right)dt + B\left(t, X\left(t\right)\right)dW\left(t\right) \tag{A.3}$$

given in differential form. Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an m-dimensional $\mathbb{F}$-Wiener process W and initial condition ξ, $X = \left(X\left(t\right)\right)_{0\le t\le T} \in \mathbb{R}^k$ is said to be a strong solution to (A.3) if X is $\mathbb{F}$-adapted, $X\left(0\right) = \xi$ $\mathbb{P}$-a.s.,

$\mathbb{P}\left[\int_0^t \left\{\|A_i(s,X(s))\| + \sigma_{ij}^2(s,X(s))\right\} ds < \infty\right] = 1 \; \forall i,\, j,\, t$, and the integral version of (A.3) satisfies

$$X(t) = X(0) + \int_0^t A(s, X(s))\, ds + \int_0^t B(s, X(s))\, dW(s), \quad \mathbb{P} - a.s.$$

For existence and uniqueness of a strong solution to (A.3) we will need the following assumptions on the coefficient functions

$(P1)$ local Lipschitz condition: $\forall\, n \in \mathbb{N}\, \exists\, c_n > 0$ such that if $|x| < n$ and $|y| < n$ it holds

$$\|A(t,x) - A(t,y)\| + \|B(t,x) - B(t,y)\| \leq c_n \|x - y\|,$$

$(P2)$ global Lipschitz condition: $\exists\, c > 0$ such that

$$\|A(t,x) - A(t,y)\| + \|B(t,x) - B(t,y)\| \leq c \|x - y\|,$$

$(P3)$ linear growth condition: $\exists\, c > 0$ such that

$$\|A(t,x)\| + \|B(t,x)\| \leq c(1 + \|x\|).$$

Further define $\|M\|^2 := \sum_{i=1}^m \sum_{j=1}^n m_{ij}^2$ for $M = (m_{ij})_{1\leq i\leq m, 1\leq j\leq n} \in \mathbb{R}^{m\times n}$.

Theorem 76 *(Existence) Assume that $(P2)$ and $(P3)$ hold true. Suppose further the initial condition $X(0)$ satisfies $\mathbb{E}\left[\|X(0)\|^2\right] < \infty$. Then there exists a strong solution to (A.3). Further the solution is bounded by $E\left[\|X(t)\|^2\right] < K\left(1 + \|X(0)\|^2\right)\exp(Kt)$.*

Theorem 77 *(Uniqueness) Assume that A and B satisfy $(P1)$. Then a solution to equation (A.3) is unique.*

A.10. Hadamard product

Given two $m \times n$-matrices $A = (a)_{ij}$, $B = (b)_{ij}$ the Hadamard product, as a component-by-component multiplication, is defined by

$$A \circ B = (ab)_{ij}.$$

A.11. Convergent Edgeworth type expansions

Let p_M be the density of the square-root scaled sum:

$$\frac{\Delta_1 + \ldots + \Delta_M}{\sqrt{M}},$$

where $\Delta_1, \ldots, \Delta_M$ are i.i.d. with $\mathtt{E}[\Delta_m] = 0$ and $\mathtt{Var}[\Delta_m] = 1$, $m = 1, ..., M$. The density p_M has a formal representation:

$$p_M(z) = \phi(z) \left[\sum_{j=0}^{\infty} \frac{h_j(z)\Gamma_{j,M}}{j!} \right]$$

with

$$h_j(z) = (-1)^j \left[\frac{d^j}{dz^j} \exp\left(-z^2/2\right) \right] \exp\left(z^2/2\right).$$

The coefficients $\Gamma_{j,M}$ are found from

$$\exp\left(\sum_{j=1}^{\infty} (\kappa_{j,M} - \alpha_j)\beta^j/j! \right) = \sum_{j=1}^{\infty} \Gamma_{j,M}\beta^j/j!,$$

where $\kappa_{j,M}$ are the cumulants of the distribution due to p_M and α_j are the cumulants of the standard normal distribution. It is clear that

$$\Gamma_{0,M} = 1$$

and that

$$\Gamma_{n,M} = \sum_{k=1}^{n} \frac{k}{n} \Gamma_{n-k,M}(\kappa_{k,M} - \alpha_k)$$

for $n > 0$. Note that $\alpha_1 = \kappa_{1,M}$ and $\alpha_k = 0$ for $k > 1$. Hence $\Gamma_{1,M} = \Gamma_{2,M} = 0$ and

$$\Gamma_{n,M} = \sum_{k=3}^{n} \frac{k}{n} \Gamma_{n-k,M}\kappa_{k,M}$$

for $n > 2$.

Lemma 78 *Let the random variable Δ_1 be bounded, i.e., $|\Delta_1| < A$ a.s., then*

$$|\Gamma_{n,M}| \leq \frac{C^n}{\sqrt{M}}$$

for some constant C depending on A.

Proof. First note that $\Gamma_{3,M} = \kappa_{3,M}$ and

$$|\kappa_{k,M}| \leq C^k M^{1-k/2}, \quad k \in \mathbb{N},$$

for some constant C depending on A. Assume that the statement is proved for all $n \leq n_0$. Then

$$
\begin{aligned}
|\Gamma_{n_0+1,M}| &\leq \frac{C^{n_0+1}}{\sqrt{M}} \sum_{k=3}^{n_0+1} \frac{k}{n_0+1} M^{1-k/2} \\
&\leq \frac{C^{n_0+1}}{\sqrt{M}} \sum_{k=3}^{n_0+1} \frac{k}{n_0+1} M^{-k/6} \leq \frac{C^{n_0+1}}{\sqrt{M}}
\end{aligned}
$$

for M large enough. ■

Since

$$|h_j(z)| \leq B^j |z|^j$$

for some $B > 0$, it holds

$$p_M(z) = \phi(z) \left[1 + \frac{D_M(z)}{\sqrt{M}} \right],$$

where

$$
\begin{aligned}
|D_M(z)| &\leq \left| \sum_{j=3}^{\infty} \frac{h_j(z)\sqrt{M}\Gamma_{j,M}}{j!} \right| \leq \left| \sum_{j=3}^{\infty} \frac{|z|^j \, (BC)^j}{j!} \right| \\
&\leq \exp(BC|z|),
\end{aligned}
$$

which would imply (4.34).

Bibliography

Andersen, L. (2000). A simple approach to the pricing of Bermudan swaptions in the multifactor LIBOR market model. *Journal of Computational Finance,* **3**, 2, 5-32.

Andersen, L. (2007). Efficient simulation of the Heston stochastic volatility model. Working paper, Bank of America.

Andersen, L. and J. Andreasen (2000). Volatility skews and extensions of the LIBOR market model. *Applied Mathematical Finance,* **7**, 1, 1-32.

Andersen, L. and M. Broadie (2004). A primal-dual simulation algorithm for pricing multidimensional American options. *Management Sciences*, **50**(9), 1222-1234.

Andersen, L. and R. Brotherton-Ratcliffe (2001). Extended Libor Market Models with Stochastic Volatility. Working paper, Gen Re Securities.

Andersen, L. B. G. and V. V. Piterbarg(2010). *Interest Rate Modeling. Volume 2: Term Structure Models.* Atlantic Financial Press.

Artzner, P., F. Delbaen, J. Eber and D. Heath (1999). Coherent measures of risk. *Mathematical finance*, **9**(3), 203–228.

Bally, V. and G. Pages (2003). A quantization algorithm for solving multidimensional discrete optimal stopping problem. *Bernoulli*, **9**(6), 1003-1049.

Barone-Adesi, G. and R. Whaley (1987). Efficient analytic approximation of American option values. *The Journal of Finance*, **42**, 301–320.

Barrieu, P. and N. El Karoui (2009). Pricing, hedging and optimally designing derivatives via minimization of risk measures. *René Carmona (ed.), Indifference Pricing: Theory and Applications. Princeton University Press, Princeton*, 77–146.

Bayraktar, E., I. Karatzas and S. Yao (2010). Optimal stopping for dynamic convex risk measures. *Illinois Journal of Mathematics*, **54**, no. 3, 1025–1067.

Bayraktar, E. and S. Yao (2011).Optimal stopping for non-linear expectations—Part I. *Stochastic Processes and their Applications*, **121**, (2), 185–211.

Becherer, D. (2009). From bounds on optimal growth towards a theory of good-deal hedging. *Albecher, H., Runggaldier, W., Schachermayer, W. (eds.) Advanced Financial Modelling, de Gruyter, Berlin*, 27–52.

Belomestny, D., C. Bender and J. Schoenmakers (2009). True upper bounds for Bermudam products via non-nested Monte Carlo. *Mathematical Finance*, **19**, (1), 53–71.

Belomestny, D., F. Dickmann and T. Nagapetyan (2013). Pricing American options via multi-level approximation methods. arXiv: 1303.1334.

Belomestny, D., M. Ladkau and J. Schoenmakers (2012). Multilevel simulation based policy iteration for optimal stopping–convergence and complexity. Tentatively accept by *SIAM/ASA Journal on Uncertainty Quantification.*

Belomestny, D., M. Ladkau and J. Schoenmakers (2012a). Tight bounds for American options via multilevel Mont Carlo. *Proceedings of the 2012 Winter Simulation Conference*, no. 31.

Belomestny, D., S. Mathew and J. Schoenmakers (2011). Multiple stochastic volatility extension of the Libor market model and its implementation. *Monte Carlo Methods Appl.*, 15, no. 4, 285–310.

Belomestny, D. and J. Schoenmakers (2006). *A Jump-Diffusion Libor Model and its Robust Calibration*, *Quant. Finance,* **11**, pp. 529–546.

Belomestny, D. and J. Schoenmakers (2011). Multilevel dual approach for pricing American style derivatives. WIAS preprint 1647. Revised version with third author F. Dickmann to appear in Finance and Stochastics.

Bender, C., A. Kolodko, J. Schoenmakers (2008). Enhanced policy iteration for American options via scenario selection. Quant. Finance **8**, No. 2, 135–146.

Bender, C. and J. Schoenmakers (2006). An iterative algorithm for multiple stopping: Convergence and stability. Advances in Appl. Prob., **38**, Number 3, pp. 729–749.

Benhamou, E., E. Gobet and M. Miri (2010). Time dependent Heston model. *SIAM Journal on Financial Mathematics,* **1**, pp.289-325

Beněs, V. E. (1970). Existence of Optimal Strategies Based on Specified Information, for a Class of Stochastic Decision Problems. *SIAM Journal of Control*, **8** (2), 179–Â–188.

Ben-Tal, A. (1985). The entropic penalty approach to stochastic programming. *Mathematics of Operations Research*, **10** (2), 224–240.

Ben-Tal, A. and M. Teboulle(1987). Penalty functions and duality in stochastic programming via ϕ-divergence functionals. *Mathematics of Operations Research*, **12** (2), 224–240.

Ben-Tal, A. and M. Teboulle(2007). An old-new concept of convex risk measures: The optimized certainty equivalent. *Mathematical Finance*, **12** (4), 449–476.

Ben-Tal, A., D. Bertsimas and D. Brown(2010). A soft robust model for optimization under ambiguity. *Operations Research*, **58** (4-2), 1220–1234.

Beskos, A. and Roberts, G. (2005). Exact simulation of diffusions. *Annals of Applied Probability*, **15**, 2422–2444.

Björk, T. and I. Slinko (2006).Towards a general theory of good-deal bounds. *Review of Finance*, **10** (2), 221–260.

Bouchard, B. and R. Elie (2008). Discrete-time approximation of decoupled Forward–Backward SDE with jumps. *Stochastic Processes and their Applications*, **118** (1), 53–75.

Brace, A., D. Gatarek and M. Musiela (1997). The Market Model of Interest Rate Dynamics. *Mathematical Finance*, **7** (2), 127–155.

Brennan, M. and E. Schwartz (1985). Evaluating natural resource investments. *Journal of Business*, **58**(2), 135–157.

Briand, P. and Confortola, F. (2008). Quadratic BSDEs with random terminal time and elliptic PDEs in infinite dimension. *Electronic Journal of Probability,* **13**, 54, 1529-1561.

Briand, P., B. Delyon and J. Mémin (2002). On the robustness of backward stochastic differential equations. *Stochastic Processes and their Applications*, **97**(2), 229–253.

Briand, P. and Hu, Y. (1998). Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDE. *Journal of Functional Analysis,* **155**, 2, 455-494.

Brigo, D. and F. Mercurio (2001). *Interest rate models—theory and practice.* Springer Finance. Springer-Verlag, Berlin.

Broadie, M. and P. Glasserman (2004). A stochastic mesh method for pricing high-dimensional American options. *Journal of Computational Finance*, **7**(4), 35–72.

Broadie, M., P. Glasserman and Z. Ha (2000). Pricing American options by simulations using a stochastic mesh with optimized weights. Probabilistic Constrained Optimization (ed. S. Uryasev): Methodology and Applications. Norwell: Kluwer Publishers, 35–50.

Broadie, M. and O. Kaya (2006). Exact simulation of stochastic volatility and other affine jump diffusion processes. *Operations Research*, **54**, 217–231.

de Bruijn, N.G. (1981). Asymptotic methods in analysis. Dover Publications, Inc. New York

Bujok, K. and B. Hambly, C. Reisinger (2012). Multilevel Simulation of Large Particle Systems and Application to Basket Credit Derivatives. Working paper.

Carr, P. and D. Madan (1999). Option Valuation Using the Fast Fourier Transform. *Journal of Computational Finance*, **2**, 61–74.

Carriere, J. (1996). Valuation of early-exercise price of options using simulations and nonparametric regression. *Insurance: Mathematics and Economics*, **19**, 19-30.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci and L. Montrucchio (2011). Uncertainty averse preferences. *Journal of Economic Theory* **146**, (4), 1275–1330.

Chen, Z. and L.G. Epstein (2002). Ambiguity, risk, and asset returns in continuous time. *Econometrica* , **70**, 1403–1443.

Chen, N. and Z. Huang (2014). Localization and exact simulation of Brownian motion driven stochastic differential equations. *Mathematics of Operations Research*, forthcoming.

Cheng, X. and F. Riedel (2013). Optimal stopping under ambiguity in continuous time, *Mathematics and Financial Economics* **7**, (1), 29–68.

Cheridito, P., F. Delbaen and M. Kupper (2006). Coherent and convex monetary risk measures for unbounded càdlàg processes. *Finance and Stochastics*, **10**, 427–448.

Choi, S., A. Ruszczyński and Y. Zhao (2011). A multiproduct risk-averse newsvendor with law-invariant coherent measures of risk. *Operations Research* **59**, (2), 346–364.

Clément, E., D. Lamberton and P. Protter (2002). An analysis of a least squares regression method for American option pricing. *Finance and Stochastics* **6**, (4), 449–471.

Cochrane, J. and J. Saá-Requejo (2000). Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets. *Journal of Political Economy* **108**, (1), 79–119.

Conrad, J. (1997). Global warming: when to bite the bullet. *Land Economics* 164-173.

Cont, R. and R. Tankov (2004). Financial Modelling with Jump Processes. *Chapmal & Hall*

Cox, J., J. Ingersoll and S.A. Ross (1985). A theory of the term structure of interest rates. *Econometrica* **53**, (2), 385–407.

Csiszár, I. (1975). *i*-divergence geometry of probability distributions and minimization problems. *Annals of Probability* **3**, (1), 146–158.

Darling, R. W. R. and Pardoux, E. (1997). Backward SDE with random terminal time and applicatoins to semilinear elliptic PDE. *The Annals of Probability,* **25**, 3, 1011-1543.

Dasci A. and G: Laporte (2005). A continuous model for multistore competitive location. *Operations Research*, **53**, (2), 263-280.

Davis, M.H.A. and I. Karatzas (1994). A Deterministic Approach to Optimal Stopping. In: Probability, Statistics and Optimisation (ed. F.P. Kelly). New York Chichester: John Wiley & Sons Ltd., 455–466

Dayanik, S. and I. Karatzas (2003). On the optimal stopping problem for one-dimensional diffusions. *Stochastic Processes and their Applications*, **107**, (2), 173–212.

Delbaen, F., S. P. and E. R. Gianin (2010). Representation of the penalty term of dynamic concave utilities. *Finance and Stochastics*, **14**, 449–472.

Dentcheva, D., S. Penev and A. Ruszczynski (2010). Kusuoka representation of higher order dual risk measures. *Annals of Operations Research*, **181**, (1), 325–335.

Dharma Kwon, H. (2010). Invest or exit? Optimal decisions in the face of a declining profit stream. *Operations Research*, **58**, (3), 638–649.

Dixit, A. (1989). Entry and exit decisions under uncertainty. *Journal of Political Economy*, 620–638.

Dixit, A. and R.S.Pindyck (1996). *Investment under uncertainty.* Princeton University Press, Princeton.

Dixit, A., R. Pindyck and A. Graham (1996). Investment under uncertainty. *Resources Policy*, **22**, (3), 217.

Duffie, D. (1992). *Dynamic asset pricing theory*, Princeton University Press.

Duffie, D. and L.G. Epstein (1992). Stochastic differential utility. *Econometrica*, **60** 353–394.

Eberlein, E. and F. Özkan (2005). The Lévy Libor model. *Finance Stoch.*, **97**, (3), 327–348.

Eisenreich, G. (1980). *Lineare Algebra und analytische Geometrie*. Akademie Verlag.

Ellsberg, D. (1961). Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics*, **75**, (4), 643–669.

Föllmer, H. and A. Schied (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, **6**, (4), 429-447.

Föllmer, H. and A. Schied (2004). *Stochastic Finance*, Walter de Gruyter & Co., Berlin.

Frittelli, M. and E. R. Gianin (2002). Putting order in risk measures. *Journal of Banking & Finance*, **26**, (7), 1473–1486.

Giesecke, K. and D. Smelov (2014). Exact sampling of jump-diffusions. *Operations Research*, forthcoming.

Gikham, I. I. and A. V. Skorokhod (1974). *The theory of stochastic processes III.* Springer.

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, **18**, (2), 141–153.

Giles, M.B. (2008). Multilevel Monte Carlo path simulation. *Operations Research*, **56**, (3), 607–617.

Glasserman, P. (2004). *Monte Carlo methods in financial engineering.* Applications of Mathematics (New York), **53**. Stochastic Modelling and Applied Probability. Springer-Verlag, New York.

Glasserman, P. and S.G. Kou (2003). The term structure of simple forward rates with jump risk. *Mathematical Finance*, **13**, (3), 383–410.

Glasserman, P. and X. Xu (2013). Robust portfolio control with stochastic factor dynamics. *Operations Research*, **61**, 874–893.

Goovaerts, M.J., R. Kaas and R.J.A. Laeven (2011). Worst case risk measurement: Back to the future? *Insurance: Mathematics and Economics* , **49**, 380–392.

Guo, X. and H. Pham (2005). Optimal partially reversible investment with entry decision and general production function. *Stochastic Processes and their Applications*, **115**, (5), 705–736.

Hagan, P. and A. Lesniewski (2008). LIBOR market model with SABR style stochastic volatility. Working paper.

Hansen, L. and R. Jagannathan (1991). Implications of Security Market Data for Models of Dynamic Economies. *Journal of Political Economy*, **99**, (2), 225–262.

Hansen, L. and T. Sargent (2001). Robust control and model uncertainty. *American Economic Review*, **91**, (2), 60–66.

Hansen, L. and T. Sargent (2007). Robustness. *Princeton University Press, Princeton.*

Haugh, M. and L. Kogan (2004). Pricing American options: a duality approach. *Operations Research*, **52**, (2), 258–270.

Heath, D. and H. Ku (2004). Pareto equilibria with coherent measures of risk. *Mathematical Finance* , **14**, (2)163–172.

Heinrich, S. (2001). Multilevel Monte Carlo Methods. *Lecture Notes in Computer Science, Springer*, **2179**, 58–67.

Henderson, V. and D. Hobson (2013). Risk aversion, indivisible timing options and gambling. *Operations Research*, **61**, 126–137.

Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, **6**, (2), 327-343.

Howard, R. (1960). Dynamic programming and Markov processes. Cambridge, Massachusetts: MIT Press.

Huber, P. (1981). Robust Statistics. *Wiley, New York.*

JS03Jacod, J. and A. N. Shiryaev (2003). Limit theoremsfor stochastic processes. *Springer.*

Jamshidian, F. (1997). LIBOR and swap market models and measures. *Finance and Stochastics*, **1**, 293–330.

Jamshidian, F. (2001). LIBOR Market Model with Semimartingales, in "Option Pricing, Interest Rates and Risk Management", Cambridge Univ.

Jiang, L. (2008). Convexity, translation invariance and subadditivity for g-expectations and related risk measures. *The Annals of Applied Probability*, **18**, (1), 245–258.

Johnson, H. (1987). Options on the maximum or the minimum of several assets. Journal of Financial and Quantitative Analysis 22, 227-83.

Joshi, M. (2003). Rapid Computation of Drifts in a Reduced Factor LIBOR Market Model. *Wilmott Magazine*, May 84-85.

Joshi, M. and R. Rebonato (2001). A stochastic-volatility, displaced-diffusion extension of the LIBOR market model. Working paper, Royal Bank of Scotland.

Kahl, C. and P. Jackel (2005). Fast strong approximation Monte-Carlo schemes for stochastic volatility models. *Working Paper, ABN AMRO and University of Wuppertal.*

Karatzas, I. and S. E. Shreve (1991). Brownian motion and stochastic calculus. *Springer.*

Karatzas, I. and S. Shreve (1998). Methods of mathematical finance. *Springer*, **39**.

Kharroubi, I. and T. Lim (2012). A decomposition approach for the discrete-time approximation of BSDEs with a jump II: the quadratic case. *arXiv:1211.6231*, **27**, (1).

Klöppel, S. and M. Schweizer (2007). Dynamic indifference valuation via convex risk measures. *Mathematical Finance*, **17**, (4), 599–627.

Kolodko, A. and J. Schoenmakers (2006). Iterative construction of the optimal Bermudan stopping time. *Finance and Stochastics*, **10**, 27–49.

Knight, F.H.(1921). *Risk, Uncertainty, and Profit.* Houghton Mifflin Co., Boston.

Krätschmer, V., M. Ladkau, R. Laeven, J. Schoenmakers and M. Stadje. (2014). Robust optimal stopping. *Working paper.*

Krätschmer, V., and J. Schoenmakers (2010). Representations for optimal stopping under dynamic monetary utility functionals. *SIAM Journal on Financial Mathematics*, **1**, (1) , 811–832.

Ladkau, M., J. Schoenmakers and J. Zhang (2013). Libor model with expiry-wise stochastic volatility and displacement. *Int. J. of Portfolio Analysis and Management*, **1**, (3) , 224–249.

Laeven, R. and M. Stadje (2013). Entropy coherent and entropy convex measures of risk. *Mathematics of Operations Research*, **38**, (2) , 265–293.

Laeven, R. and M. Stadje (2013a). Robust portfolio choice and indifference valuation. *forthcoming: Mathematics of Operations Research.*

Lee, S.-H. and P. W. Glynn (2003). Computing the distribution function of a conditional expectation via Monte Carlo: Discrete conditioning spaces. ACM Transactions on Modeling and Computer Simulation, 13(3): 238–258, July 2003.

Lesnevski, V., B. Nelson and J. Staum(2007). Simulation of coherent risk measures based on generalized scenarios. *Management Science*, **53**, (11), 1756–1769.

Longstaff, F.A. and E.S. Schwartz (2001). Valuing American options by simulation: a simple least-squares approach. *Review of Financial Studies*, **14**, 113–147.

Lord R. and C. Kahl (2010). Complex logarithms in Heston-like models. *Math. Fin.*, **20**, Issue 4, 671–694.

Lord, R., R. Koekkoek and D. van Dijk (2010). A comparison of biased simulation schemes for stochastic volatility models. *Quant. Finance*, **17**, (4), 373–376.

Maccheroni, F., M. Marinacci and A. Rustichini (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, **74**, (6), 1447–1498.

McDonald, R. and D. Siegel (1986). The value of waiting to invest. *The Quarterly Journal of Economics*, **101**, (4), 707–727.

Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, **3**, (1), 125–144.

Miltersen, K., K. Sandmann and D. Sondermann (1997). Closed-form solutions for term structure derivatives with lognormal interest rates. *Journal of Finance*, 409-430.

Morini, M. and F. Mercurio (2007). No-arbitrage dynamics for a tractable SABR term structure LIBOR model, preprint.

Øksendal, B. (2003). Stochastic Differential Equations. *Springer.*

Øksendal, A. (2000). Irreversible investment problems. *Finance and Stochastics*, **4**, (2), 223–250.

Øksendal, B. and K. Reikvam (1998). Viscocity solutions of optimal stopping problems. *Stochastics and Stochastics Reports*, **62**, (3-4), 285–301.

Øksendal, B. and A. Sulem (2007). Applied stochastic control of jump diffusions, second ed. *Universitext. Springer, Berlin*, **62**, (3-4), 285–301.

Øksendal, B., A. Sulem and T. Zhang (2013). Singular control and optimal stopping of SPDEs, and backward SPDEs with reflection. *Mathematics of Operations Research*, forthcoming.

Papapantoleon, A., J. Schoenmakers and D. Skovmand (2011). Efficient and accurate log-Levy approximations to Levy driven LIBOR models. *J. of Computational Finance* (to appear).

Pardoux, E. and S. Peng (1990). Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, **14**, (1), 55–61.

Peskir, G. and A. N. Shiryaev (2006). Optimal stopping and free-boundary problems. *Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel.*

Peng, S. (1997). Backward SDE and related g-expectations. *El Karoui, N. and Mazliak, L. (eds.), Backward stochastic differential equations, Pitman Res. Notes Math. Ser. Vol. 364, Longman, Harlow*, 141–159.

Peng, S. (1997). Nonlinear Expectations, Nonlinear Evaluations and Risk Measures. *Lecture Notes in Mathematics*, **1856** 165-253.

Philpott, A., V. de Matos and E. Finardi (2013). On solving multistage stochastic programs with coherent risk measures. *Operations Research, forthcoming.*

Pindyck, R. (1986). Irreversible investment, capacity choice, and the value of the firm. *Mimeo.*

Piterbarg, V. (2004). A stochastic volatility forward Libor model with a term structure of volatility smiles. *SSRN Working Paper.*

Platen, E. (1999). An introduction to numerical methods for stochastic differential equations. *School of Finance and Economics, University of Technology, Sydney.*

Puterman, M. (1994). Markov decision processes. *NewYork: Wiley.*

Reiß, O., J. Schoenmakers and M. Schweizer (2007). From structural assumptions to a link between assets and interest rates. *J. of Econ. Dyn. and Control.*

Riedel, F. (2004). Dynamic coherent risk measures. *Stochastic processes and their Applications*, **112**, (2), 185–200.

Riedel, F. (2009). Optimal stopping with multiple priors. *Econometrica*, **77**, (3), 857–908.

Rogers, L.C.G. (2002). Monte Carlo valuation of American options. *Mathematical Finance*, **12**, 271-286.

Ruszczyński, A. (2010). Risk-averse
dynamic programming for Markov decision processesRisk-averse dynamic programming for Markov decision processes. *Mathematical Programming Series B*, **125**, (2), 235–261.

Ruszczyński, A. and A. Shapiro (2006). Optimization of convex risk functions. *Mathematics of Operations Research*, **31**, (3), 433–452.

Ruszczyński, A. and A. Shapiro (2006). Conditional risk mappings. *Mathematics of Operations Research*, **31**, (3), 544–561.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, **57**, (3), 571–587.

Schoenmakers, J. (2005). *Robust Libor Modelling and Pricing of Derivative Products.* BocaRaton London NewYork Singapore: Chapman & Hall – CRC Press.

Schoenmakers, J., J. Zhang and J. Huang (2013). Optimal dual martingales, their analysis and application to new algorithms for Bermudan products. To appear in *SIAM Journal on Financial Mathematics.*

Shapiro, A., D. Dentcheva and A. Ruszczynski (2009). Lectures on stochastic programming: modeling and theory. *SIAM*, **9**.

Snell, L.J. (1952). Applications of martingale system theorems. *Trans. Amer. Math. Soc.*, **73**, 293–312.

Stroock, D. W. and S. R. S. Varadhan (1979). Multidimensional diffusion processes. *Springer.*

Tang, S. and X. Li (1994).Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM Journal of Control and Optimization*, **32**, 1447–1475.

Tang, S. and W. Wei (2012). Representation of dynamic time-consistent convex risk measures with jumps. *Risk and Decision Analysis*, **3**, (3), 167–190.

Tekaya, W., S. A. S. M. P. and J. P. da Costa (2013). Worst-case-expectation approach to optimization under uncertainty. To appear in: *Operations Research.*

Tsitsiklis, J. and B. Van Roy (2000). Regression methods for pricing complex American style options. *IEEE Trans Neural. Net.*, **12**, (4), 694–703.

Wald, A. (1950). Statistical Decision Functions. *Wiley.*

Wu, L. and F. Zhang (2006). Libor Market Model with Stochastic Volatility. *Journal of Industrial and Management Optimization*, **2**, 199–207.

Wu, L. and F. Zhang (2008). Fast swaption pricing under the market model with a square-root volatility process. *Quantitative Finance,* **8**, (2), 163–180.

Zhu, H., F. Ye and E. Zhou (2013). True martingales for upper bounds on Bermudan option prices under jump-diffusion processes. *Proceedings of the 2013 Winter Simulation Conference.*

List of Symblos

$:=$,	defined as
$a \in A$,	a is an element of A
$a \notin A$,	a is no element of A
$A \cup B$,	union of A and B
$A \cap B$,	intersection of A and B
$\varnothing$,	empty set
$A \subseteq B$,	A is a subset of B
$A \subset B$,	A is a real subset of B, $A \neq B$
$\mathbb{R}^m$,	space of m-dimensional vectors with real valued entries
$\mathbb{N}$	set of natural numers including 0
a^T,	transposed of a matrix
$\lVert a \rVert$,	norm of a, $\lVert a \rVert^2 := \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2$ if $a \in \mathbb{R}^{m \times n}$
$\lvert a \rvert$,	$\lVert a \rVert = \lvert a \rvert$, for a being a scalar
$span\,(S)$,	subspace spanned by set of vectors S, given by all possible linear combinations of S
$span^{\perp}\,(S)$,	orthogonal complement of S, given a vector space V
dim,	dimension of a space
$rg\,(A)$,	rank of matrix A
$I_{\{A\}}$,	indicator function, $I_{\{A\}} = \begin{cases} 1, & A = true \\ 0, & A = false \end{cases}$
$\phi\,(x)$,	density function of standard normal distribution
$\Phi\,(x)$,	cumulative density function of standard normal distribution

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 20.01.2015 Marcel Ladkau